Quantum finance Hamiltonian for coupon bond European and barrier options

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Coupon bond European and barrier options are financial derivatives that can be analyzed in the Hamiltonian formulation of quantum finance. Forward interest rates are modeled as a two-dimensional quantum field theory and its Hamiltonian and state space is defined. European and barrier options are realized as transition amplitudes of the time integrated Hamiltonian operator. The double barrier option for a financial instrument is "knocked out" (terminated with zero value) if the price of the underlying instrument exceeds or falls below preset limits; the barrier option is realized by imposing boundary conditions on the eigenfunctions of the forward interest rates' Hamiltonian. The price of the European coupon bond option and the zero coupon bond barrier option are calculated. It is shown that, is general, the constraint function for a coupon bond barrier option can—to a good approximation—be linearized. A calculation using an overcomplete set of eigenfunctions yields an approximate price for the coupon bond barrier option, which is given in the form of an integral of a factor that results from the barrier condition times another factor that arises from the payoff function.

PACS number(s): 89.65.−s

I. INTRODUCTION

European and barrier options are widely used in finance for both stocks (equity) and bonds (debt). It is estimated that by early 2008, the notional value of the global derivatives markets (futures, options, swaps, etc.) exceeded US$500 trillion, with over 70% being interest rate derivatives. European options form the backbone of interest rate derivatives. Interest rate barrier options have been increasing in importance and are being widely used as hedging instruments for risk management strategies.

Quantum finance [1] refers to the application of the theoretical and mathematical formalism of quantum mechanics and quantum field theory to problems arising in finance. The theory and application of coupon bond and interest rate options has been studied in some detail in Ref. [2] using the approach of quantum finance. The Hamiltonian formulation of quantum finance is useful for studying a special class of options, namely, European and barrier options. The case of equity options using the Hamiltonian approach has been discussed in Ref. [3] and, in this paper, the Hamiltonian approach is extended to the more complex case of European and barrier options for coupon bonds. Interest rate options are a special case of coupon bond options [4].

Forward interest rates $f(t,x)$ are the interest rates, fixed at time t, for an instantaneous loan at future times $x > t$; $f(t,x)$ has the dimensions of 1/time. Let present time be denoted by t_0; since the future is unknown, all forward interest rates $f(t,x)$, for $t > t_0$, are taken to be stochastic quantities. The prices of all interest rate options are given by an appropriate expectation value, with the average being obtained by integrating the classical stochastic field $f(t,x)$ over all it's possible values. This averaging procedure is formally equivalent to how expectation values are calculated in quantum field theory defined for imaginary (Euclidean) time; hence, in ef-

fect, in quantum finance $f(t,x)$ is taken to be mathematically equivalent to a two-dimensional Euclidean quantum field. The volatility $\sigma^2(t,x)$ of the forward interest rates $f(t,x)$ is a small quantity, of the order of $10^{-2}/\mathrm{yr}^2$, and hence provides a small parameter for obtaining an approximate value for the option price.

The paper is organized as follows. In Sec. II the Hamiltonian and state space of the forward interest rates are defined. A new set of "bond" variables are introduced that are useful for the study of coupon bonds. In Sec. III the European and double barrier options are defined and reexpressed in a form suitable for an approximate compuation of their price. The forward bond numeraire is introduced for simplifying the calculation. In Sec. IV the Hamiltonian, state space and the evolution operator for the forward interest rates are discussed. In Sec. V the approximate European coupon bond option price—derived earlier using path integration [5]—is rederived using the Hamiltonian formulation. In Sec. VI the price of barrier option is defined using the concept of state space and matrix elements of the evolution operator. In Sec. VII the special case of the barrier option's price for the zero coupon bond is derived. In Sec. VIII a consistent linearization of the coupon bond payoff function is discussed. In Sec. IX an overcomplete set of eigenfunctions, necessary for computing the approximate coupon bond barrier option price, is introduced and Sec. X yields an approximate price of the coupon bond barrier option. In Sec. XI two limiting cases of the coupon bond barrier option, namely the European option and the zero coupon bond option, are obtained. In Sec. XII some conclusions are drawn. The more technical aspects of the derivations are discussed in Appendixes A and B.

II. HAMILTONIAN AND STATE SPACE

The Hamiltonian and the state space (of the forward interest rates) are two independent ingredients of a quantum theory; taken together they reproduce the (forward interest
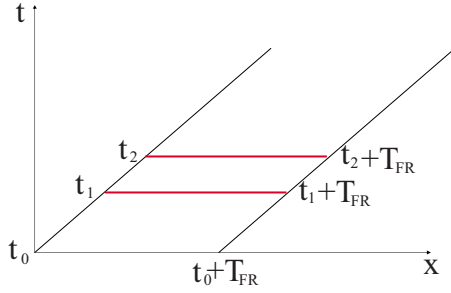
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FIG. 1. (Color online) The domain of the state space of the forward interest rates. For fixed time t, the state space $\mathcal{V}_t$ consists of all possible financial instruments (functions) of instantaneous forward interest rates $f(t,x)$ with $x \in [t, t+T_{FR}]$. In the figure the state space $\mathcal{V}_t$ are indicated for two distinct calendar times t_1 and t_2, with a distinct forward interest rates $f(t,x)$ for each point on the line.

rates') Lagrangian. The essential features of the forward interest rates' Hamiltonian and state space are reviewed [1]. The state space of a field theory, similar to all quantum systems, is a linear vector space—denoted by $\mathcal{V}$—that consists of functionals of the field configurations at some fixed time t. The dual space of $\mathcal{V}$—denoted by $\mathcal{V}_{dual}$—consists of all linear mappings from $\mathcal{V}$ to the complex numbers, and is also a linear vector space. The Hamiltonian $\mathcal{H}$ is an operator—the quantum analog of energy—that is an element of the tensor product space $\mathcal{V} \otimes \mathcal{V}_{dual}$ and maps the state space to itself, that is $\mathcal{H}: \mathcal{V} \rightarrow \mathcal{V}$.

The forward interest rates $f(t,x)$ are defined only for future time $x > t$ and upto a maximum future time T_{FR}, which is usually 30 years; that is, $t < x < t+T_{FR}$. The Hamiltonian for the forward interest rates $f(t,x)$ is more complicated than what usually occurs in physics due to the nontrivial structure of the underlying trapezoidal domain $\mathcal{T}$ of the t,x space, given in Fig. 1; since $x \in [t, t+T_{FR}]$ the quantum field $f(t,x)$ exists only for future time, that is, for $x > t$. In particular, the forward interest rates' quantum field has a distinct state space $\mathcal{V}_t$ for every instant t.

The state space at time t is labeled by $\mathcal{V}_t$, and it's state vectors by $|f_t\rangle$. The elements of the state space of the forward rates $\mathcal{V}_t$ includes all possible financial instruments that are traded in the market at time t. In continuum notation, the basis states of $\mathcal{V}_t$ are tensor products over the future time x and satisfy the following completeness equation:

$$|f_t\rangle = \prod_{t \leq x \leq t+T_{FR}} |f(t,x)\rangle,$$

$$\mathcal{I}_t = \prod_{t \leq x \leq t+T_{FR}} \int_{-\infty}^{+\infty} df(t,x) |f_t\rangle\langle f_t| \equiv \int Df_t |f_t\rangle\langle f_t|. \quad (1)$$

Figure 1 shows the domain of the state space as a function of time t.

The time-dependent Hamiltonian $\mathcal{H}(t)$ is the backward Fokker-Planck Hamiltonian and propagates the interest rates backwards in time, taking the final state $|f_{final}\rangle$ given at time T_f backwards to an initial state $\langle f_{initial}|$ at the earlier time T_i. The transition amplitude Z for a time interval $[T_i, T_f]$ can be
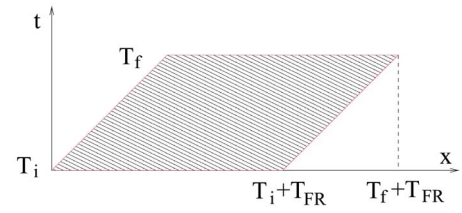

FIG. 2. (Color online) The trapezoidal domain $\mathcal{T}$ of the forward interest rates required for computing the transition amplitude $\langle f_{initial}|\mathcal{T}\{\exp - \int_{T_i}^{T_f} H(t)dt\}|f_{final}\rangle$. The composition of successive state spaces $\mathcal{V}_t$ yields the trapezoidal domain.

constructed from the Hamiltonian and state space by applying the time slicing method; since the state space and Hamiltonian are both time dependent one has to use the time-ordering operator $\mathcal{T}$ to keep track of the time dependence. The transition amplitude between a final (coordinate basis) state $|f_{final}\rangle$ at time T_f to an arbitrary initial (coordinate basis) state $\langle f_{initial}|$ at time T_i is given by

$$Z = \langle f_{initial}|T\left\{\exp - \int_{T_i}^{T_f} H(t)dt\right\}|f_{final}\rangle \quad (2)$$

Due to the time dependence of the state spaces $\mathcal{V}_t$ the forward interest rates that determine Z form a trapezoidal domain shown in Fig. 2.

The degrees of freedom $f(t,x)$ refer to time t only through the domain on which the Hamiltonian is defined. The Hamiltonian is an infinitesimal generator in time, and refers to only the instant of time at which it acts on the state space. This is the reason that in the Hamiltonian the time index t can be dropped for the variables $f(t,x)$, with $f(x), t \leq x \leq t+T_{FR}$. The Hamiltonian for forward interest rates is given by

$$\mathcal{H}(t) = -\frac{1}{2}\int_t^{t+T_{FR}} dxdx' M(x,x';t)\frac{\delta^2}{\delta f(x)\delta f(x')}$$
$$- \int_t^{t+T_{FR}} dx\,\alpha(t,x)\frac{\delta}{\delta f(x)},$$

$$M(x,x';t) = \sigma(t,x)D(x,x';t)\sigma(t,x').$$

General considerations related to the existence of a martingale measure rule out any potential terms for the forward interest rates' Hamiltonian; the entire dynamics is contained in the kinetic term, with different choices of the function $M(x,x';t)$ encoding a wide variety of forward interest rates models. It has been shown that the "stiff" propagator fits the historical market data for the forward interest rates for the U.S. dollar (called Libor) and for the Euro (called Euribor) with a root means square error of less than 1% [6]. The drift term $\alpha(t,x)$ in Hamiltonian is non-Hermitian, as is typical for the case of finance, and is fixed by the choice of the discounting factor (to be discussed in the next section).

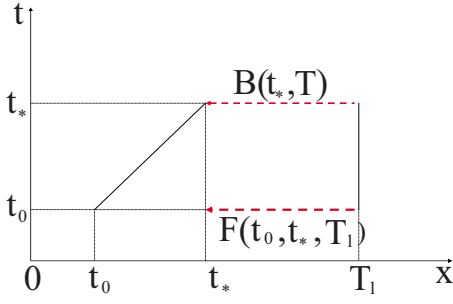The function $\sigma(t,x) > 0$ is directly related to the volatility of the forward interest rates and is given by

FIG. 3. (Color online) Figure shows the zero coupon bond $B(t_*,T_1)$ and its forward bond price at t_0, namely, $F(t_0,t_*,T_1)$.

$$E\left[\frac{\partial f(t,x)}{\partial t}\frac{\partial f(t',x)}{\partial t'}\right] - E\left[\frac{\partial f(t,x)}{\partial t}\right]E\left[\frac{\partial f(t',x)}{\partial t'}\right]$$
$$= \delta(t-t')\sigma^2(t,x),$$

where $E[\cdots]$ denotes the expectation value taken over the random forward interest rates $f(t,x)$.

The empirical value of $\sigma^2(t,x) \rightarrow \sigma^2(x-t)$ in quantum finance is equal to $E[\delta f(t,x)\delta f(t,x)]$ (with dimension of yr^{-2}), where $\delta f(t,x)$ is the change of $f(t,x)$ in one day. The empirical expectation value is obtained by averaging over the market values of $\delta f(t,x)$. The empirical volatility $\sigma^2(t,x)$ for Libor and Euribor is a small quantity, of the order of $10^{-2}/\mathrm{yr}^2$, and hence, as mentioned in the Introduction, provides a small parameter for obtaining an approximate value for the various option prices.

III. COUPON BONDS AND OPTIONS

A zero coupon bond gives a pre-determined payoff of say \$1 when it matures at some fixed time T; its price at earlier time $t < T$, denoted by $B(t,T)$, is given by discounting the payoff of \$1, paid at time T, to present time t by using the prevailing forward interest rates [9]. Discounting the \$1 payoff, paid at maturity time T, by taking infinitesimal backward time steps ϵ from T to present time t yields [10]

$$B(t,T) = e^{-\epsilon f(t,t+\epsilon)}e^{-\epsilon f(t,t+2\epsilon)}\cdots e^{-\epsilon f(t,x)}\cdots e^{-\epsilon f(t,T)}\$1$$
$$\Rightarrow B(t,T) = \exp\left\{-\int_t^T dx f(t,x)\right\}.$$

Suppose a zero coupon bond $B(t_*,T_i)$ is going to be issued at some future time $t_* > t_0$, with expiry at time T_i. $F(t_0,t_*,T_i)$ is the forward price of $B(t_*,T_i)$ that one pays at time t_0—to lock-in the delivery of the bond when it is issued at time t_*—and is given by

$$F(t_0,t_*,T_i) = \exp\left\{-\int_{t_*}^{T_i} dx f(t_0,x)\right\} = \frac{B(t_0,T)}{B(t_0,t_*)}$$

$$\text{:forward bond price.} \qquad (3)$$

Both the zero coupon bond and its forward price are shown in Fig. 3.

Consider a coupon bond on a principal L that is issued at time t_*, matures at time T and pays fixed dividends (cou-

pons) a_i at times T_i, $i=1,2,\ldots N$. The value of the coupon bond at time $t_* < T_i$ can be shown to be given by

$$\mathcal{B}(t_*,T) = \sum_{i=1}^{N} a_i B(t_*,T_i) + LB(t_*,T) = \sum_{i=1}^{N} c_i B(t_*,T_i), \quad (4)$$

where for simplicity of notation the final payment is included in the sum by setting $c_i = a_i$; $c_N = a_N + L$, and with the time of maturity of the coupon bond given by $T = T_N$.

The payoff function $\mathcal{P}(t_*) = \mathcal{P}_*$ of a coupon bond European call option maturing at time t_* and with strike price K is given by

$$\mathcal{P}_* = \left(\sum_{i=1}^{N} c_i B(t_*,T_i) - K\right)_+ = [\mathcal{B}(t_*,T) - K]_+. \quad (5)$$

Note that

$$(a-b)_+ \equiv (a-b)\Theta(a-b) \qquad (6)$$

and the Heaviside step function $\Theta(x)$ is defined by

$$\Theta(x) \equiv \begin{cases} 1, & x > 0, \\ \dfrac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \qquad (7)$$

Let $C(t_0,t_*,T,K) = C_{[t_*,T,K]}(t_0)$ be the price of a call option that one is seeking to ascertain at (present) time t_0. The payoff function is defined to be the value of the option at time t_*, namely, that $C_{[t_*,T,K]}(t_*) = \mathcal{P}_*$. A fundamental theorem of finance states that the price of an option on an underlying security is free from arbitrage opportunities only if the underlying security is modeled to have a Martingale evolution. The theory of option pricing hinges on the property of martingales. The underlying security can have a martingale evolution if it is discounted by an appropriate numeraire. One can choose the numeraire from a large collection of discounting factors. For calculating the price of interest rate options, it is very convenient to discount the future price of the option by the using the zero coupon bond $B(t_0,t_*)$ as discount factor. One then has that $C(t_0,t_*,T,K)/B(t_0,t_*)$ is a Martingale; in particular the expectation value of the future random value of a Martingale is equal to its present value. In terms of equations, the price of option at time $t_0 < t_*$ is, hence, given by the Martingale condition as follows [using $B(t_*,t_*)=1$]

$$\frac{C_{[t_*,T,K]}(t_0)}{B(t_0,t_*)} = E_{[t_0,t*]}\left[\frac{C_{[t_*,T,K]}(t_*)}{B(t_*,t_*)}\right] = E_{[t_0,t_*]}[\mathcal{P}_*]$$

$$\Rightarrow C(t_0,t_*,T,K) = B(t_0,t_*)E[\mathcal{P}_*], \qquad (8)$$

where for simplicity of notation the subscript on the expectation value $E[\cdots]$ has been dropped. Note that the discounting factor $B(t_0,t_*)$ is determined by the initial value of the forward interest rates $f(t_0,x)$ and hence is not a random quantity.

One can think of Eq. (8) as the price of a (European call) option at time $t_0 < t_*$ being given by discounting the payoff

$\mathcal{P}_*$ from time t_* to time t_0 and averaging over all the random (fluctuating) forward interest rates over future calendar time $[t_0, t_*]$—with the initial conditions specified at time t_0 by $f(t_0, x)$.

Choosing the discount factor to be $B(t_0, t_*)$ fixes the drift velocity to be the following:

$$\alpha(t,x) = \sigma(t,x) \int_{t_*}^{x} dx' D(x,x';t) \sigma(t,x') = \int_{t_*}^{x} dx' M(x,x';t).$$

(9)

Equation (9) fixes $\alpha(t,x)$ and hence completes the specification of Hamiltonian for the forward rates given in Eq. (3).

Approximate option price

The forward price of the coupon bond $\mathcal{B}(t_*)$ at earlier time $t_0 < t_*$ is given by

$$\sum_{i=1}^{N} c_i F_i, \quad F_i \equiv F(t_0, t_*, T_i) = \exp\left\{ -\int_{t_*}^{T_i} dx f(t_0, x) \right\}.$$

Since volatility σ is small, one expects that deviations of $\mathcal{B}(t_*)$ from it's initial forward price at time t_0 should be small and hence amenable to a perturbative expansion. The price of the coupon bond is consequently rewritten as

$$\sum_{i=1}^{N} c_i B(t_*, T_i) = \sum_{i=1}^{N} c_i F_i + \sum_{i=1}^{N} c_i [B(t_*, T_i) - F_i] \equiv F + V,$$

$$V = \sum_i J_i \left[\frac{B(t_*, T_i)}{F_i} - 1 \right], \quad J_i = c_i F_i.$$

(10)

A representation of the payoff function, which is useful for the option computation, is to subtract out the forward value of the coupon bond at time t_0. Using the representation of the Dirac delta function given

$$\delta(Q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta\, e^{i\eta Q}$$

(11)

yields the following expression for the payoff function:

$$\left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ = (F + V - K)_+$$

$$= \int_{-\infty}^{+\infty} dQ\, \delta(V - Q)(F + Q - K)_+$$

$$= \int_{-\infty}^{+\infty} dQ \frac{d\eta}{2\pi} e^{i\eta(V-Q)} (F + Q - K)_+$$

$$\equiv \int_{Q,\eta} e^{i\eta(V-Q)} (F + Q - K)_+$$

(12)

and hence

$$\mathcal{P}_* = \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+$$

$$\simeq \int_{Q,\eta} e^{-i\eta Q} (F + Q - K)_+$$

$$\times \left[1 + i\eta V - \frac{1}{2}\eta^2 V^2 + O(\sigma^3) \right],$$

$$V = \sum_I J_I [B(t_*, T_I) e^{f_I} - 1],$$

$$f_I = \int_{t_*}^{T_I} dx f(t_0, x), \quad J_I = c_I e^{-f_I}, \quad F = \sum_I J_I.$$

(13)

Note V is the only random quantity in the payoff function and hence yields the following expansion for the coupon bond option price:

$$\frac{C(t_0, t_*, T, K)}{B(t_0, t_*)} = E[\mathcal{P}_*]$$

$$\simeq \int_{Q,\eta} e^{-i\eta Q} (F + Q - K)_+ \left[E[1] + i\eta E[V] \right.$$

$$\left. - \frac{1}{2}\eta^2 E[V^2] \right]$$

$$\simeq \int_{Q,\eta} e^{i\eta Q} (F + Q - K)_+ \left[C_0 + i\eta C_1 \right.$$

$$\left. - \frac{1}{2}\eta^2 C_2 + O(\sigma^3) \right],$$

(14)

where, for $g_I = \int_{t_*}^{T_I} dx f(t_*, x)$, the coefficients of the expansion are given by

$$C_0 = E[1],$$

(15)

$$C_1 = E[V] = \sum_I J_I (C_I - C_0), \quad C_I = E[e^{-g_I + f_I}],$$

(16)

$$C_2 = E[V^2] = \sum_{IK} J_I J_K (C_{IK} - C_I - C_K + C_0),$$

$$C_{IK} = E[e^{-g_I - g_K + f_I + f_K}].$$

(17)

The approximate price for both the coupon bond European and barrier options are obtained by evaluating the coefficients C_0, C_I, and C_{IK}.

Equation (14) yields, for $D_1 = C_1 / C_0$ and $D_2 = C_2 / C_0$, the following approximate option price:

$$\frac{C(t_0,t_*,T,K)}{B(t_0,t_*)} \simeq C_0 \int_{Q,\eta} e^{-i\eta Q}(F+Q-K)_+$$

$$\times \exp\left\{i\eta D_1 - \frac{1}{2}\eta^2(D_2 - D_1^2)\right\} + O(\sigma^3)$$

$$= C_0 \int_{Q,\eta} e^{-i\eta Q}(F+Q+D_1-K)_+$$

$$\times \exp\left\{-\frac{1}{2}\eta^2(D_2 - D_1^2)\right\}$$

$$= \frac{1}{\sqrt{2\pi}}C_0 I(X)\sqrt{D_2 - D_1^2},$$

$$X = \frac{K-F-D_1}{\sqrt{D_2 - D_1^2}}. \tag{18}$$

The function $I(X)$ is given in terms of the error function $\Phi(u)$ as follows:

$$I(X) = \int_{-\infty}^{+\infty} dQ(Q-X)_+ e^{-(1/2)Q^2}$$

$$= e^{-(1/2)X^2} - \sqrt{\frac{\pi}{2}}X\left[1 - \Phi\left(\frac{X}{\sqrt{2}}\right)\right].$$

$$\Phi(u) = \frac{2}{\sqrt{\pi}}\int_0^u dQ e^{-Q^2}.$$

The asymptotic behavior of the error function $\Phi(u)$ yields the following limits:

$$I(X) = \begin{cases} 1 - \sqrt{\dfrac{\pi}{2}}X + O(X^2), & X \approx 0, \\[3mm] \dfrac{e^{-1/2X^2}}{X^2}\left[1 + O\left(\dfrac{1}{X^2}\right)\right], & X \gg 0. \end{cases}$$

The coupon bond option, at the money, has $F=K$; hence the option's price, close to at the money, has $X \approx 0$ and yields the following approximate price

$$C(t_0,t_*,K) \approx \frac{1}{\sqrt{2\pi}}B(t_0,t_*)C_0\sqrt{D_2 - D_1^2} - \frac{1}{2}B(t_0,t_*)C_0(K-F$$

$$- D_1) + O(X^2). \tag{19}$$

IV. HAMILTONIAN AND EVOLUTION OPERATOR FOR COUPON BONDS

In the notation of the Hamiltonian and state space, the payoff function is a state vector that is an element of the state space $\mathcal{V}_*$ at future time t_*; in terms of the coordinate basis eigenstate of the dual state space V_{dual}

$$\langle f_*| \equiv \prod_{t_* \leq x \leq t_* + T_{FR}} \langle f(x)|$$
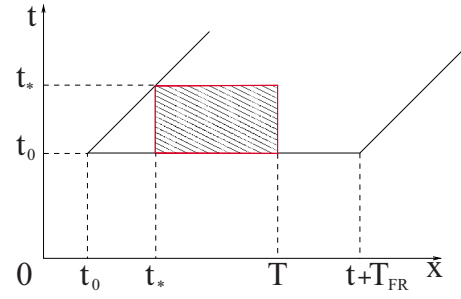
the payoff function is given by

FIG. 4. (Color online) The domain $\mathcal{R}$ of forward interest rates that determines the price of the coupon bond option when discounted by $B(t_0,t_*)$.

$$\langle f_*|\mathcal{P}_*\rangle = \mathcal{P}_*[f_*] = \left(\sum_{i=1}^N c_i B(t_*,T_i) - K\right)_+.$$

To make the content of the payoff function $\mathcal{P}_*$ more explicit, note that

$$\langle f_*|\mathcal{P}_*\rangle = \begin{cases} \left[\displaystyle\sum_{i=1}^N c_i B(t_*,T_i) - K\right]_+, & t_* \leq x \leq T, \\[3mm] 0, & x > T, \end{cases}$$

$$\equiv \mathcal{P}[f_*].$$

From above, it can be seen that the payoff function $|\mathcal{P}_*\rangle$ has nonzero components in the future direction x only in the interval $t_* \leq x \leq T$.

In the Hamiltonian formulation the option price at time t_0 is given by propagating the payoff function $|\mathcal{P}_*\rangle$, defined at time t_*, backwards in time, as given in Eq. (2), to present time t_0, discounted by the numeraire $B(t_0,t_*)$. Hence, in terms of the basis state at t_0 given by $\langle f^{(0)}|$, the European coupon option price, as shown in Fig. 2, is given by

$$C_E(t_0,t_*,T,K) = B(t_0,t_*)E\{[\mathcal{B}(t_*,T) - K]_+\}$$

$$= B(t_0,t_*)\langle f^{(0)}|\mathcal{T}\left\{\exp - \int_{t_0}^{t_*} dt H(t)\right\}|\mathcal{P}_*\rangle,$$

$$\langle f^{(0)}| \equiv \prod_{t_0 \leq x \leq t_0 + T_{FR}} \langle f(x)|. \tag{20}$$

It is shown in Appendix A that the time ordering for the Hamiltonian simplifies when $B(t_0,t_*)$ is the discounting factor; the domain of the forward interest rates required for determining the option prices simplifies into a rectangle $\mathcal{R}$ equal to $[t_0,t_*] \times [t_*,T]$ and is shown in Fig. 4. The dynamics of the option price for both the European and barrier option is now determined by a new time independent operator W that is given by the integration of the time dependent Hamiltonian. From Eq. (A3)

$$W_f = \int_{t_0}^{t_*} dt \mathcal{H}(t) = -\frac{1}{2}\int_{t_*}^{T} dx dx' M(x,x') \frac{\delta^2}{\delta f(x)\,\delta f(x')}$$

$$-\int_{t_*}^{T} dx\,\alpha(x)\frac{\delta}{\delta f(x)}, \qquad (21)$$

$$M(x,x') = \int_{t_0}^{t_*} dt M(x,x';t),$$

$$\alpha(x) = \int_{t_0}^{t_*} dt\,\alpha(t,x) = \int_{t_0}^{t_*} dt \int_{t_*}^{x} dx' M(x,x';t). \qquad (22)$$

The operator W acts on the (fixed) state space $\mathcal{V}_*$ that is spanned by the basis states $\langle f_*^{(0)}| \equiv \langle f_*| = \Pi_{t_*\leq x\leq T}\langle f(t_0,x)|$. $\mathcal{V}_*$ is a subspace of the t_0 state space $\mathcal{V}_{t_0}$; note $|\mathcal{P}_*\rangle \in \mathcal{V}_*$.

The European coupon bond option price, from Eq. (A4), is given by [11]

$$C_E(t_0,t_*,T,K) = B(t_0,t_*)\langle f_*|e^{-W_f}|\mathcal{P}_*\rangle.$$

The calculation is carried out at calendar time t_0 and all the effects coming from future calendar time from t_0 to t_* are carried by the coefficients $M(x,x')$ and $\alpha(x)$. In other words, the option price calculation is carried out in the fixed state space $\mathcal{V}_*$. W is a differential operator that contains the correlations of the forward interest rates in future time direction x.

The state vector $e^{-W_f}|\mathcal{P}_*\rangle$ is the price of the option at time t_0; the operator e^{-W_f} propagates the payoff function from future time t_* to its present value at time t_0. The operator W_f is the evolution operator and e^{-W_f} evolves the payoff state vector, defined at calendar time t_*, backwards in time.

The natural coordinates for the evolution operator in studying coupon bonds and swaptions is the integral of the forward interest rates, namely, the dimensionless bond variable $g(x)$ defined by

$$g(x) = \int_{t_*}^{x} dy f(y), \qquad \frac{\delta g(x)}{\delta f(y)} = \theta(x-y), x,y \geq t_*. \qquad (23)$$

To express W_f in terms of $\delta/\delta g(x)$ note that the chain rule of differentiation and Eq. (23) yield

$$\frac{\delta}{\delta f(y)} = \int_{y}^{T} dx \frac{\delta}{\delta g(x)}. \qquad (24)$$

From Eqs. (21) and (24) and after some simplifications

$$W_g = -\frac{1}{2}\int_{t_*}^{T} dx \int_{t_*}^{T} dx' G(x,x')\frac{\delta^2}{\delta g(x)\,\delta g(x')}$$

$$-\int_{t_*}^{T} dx\,\beta(x)\frac{\delta}{\delta g(x)},$$

$$G(x,x') = \int_{t_*}^{x} dy \int_{t_*}^{x'} dy' \int_{t_0}^{t_*} dt M(y,y';t),$$

$$\beta(x) = \int_{t_*}^{x} dy \int_{t_0}^{t_*} dt\,\alpha(t,y) = \int_{t_*}^{x} dy \int_{t_*}^{y} dy' \int_{t_0}^{t_*} dt M(y,y';t)$$

$$= \frac{1}{2}G(x,x). \qquad (25)$$

The evolution operator W_g simplifies further when it acts on only coupon bond variables. Note that

$$\frac{\delta g(x)}{\delta g(x')} = \delta(x-x'), \quad x,x \in [t_*,T]$$

and this yields, for an arbitrary function of the bond variables $\mathcal{S}[g_1,\dots,g_N] = \mathcal{S}[\mathbf{g}]$, the following:

$$\frac{\delta}{\delta g(x)}\mathcal{S}[\mathbf{g}] = \sum_i \frac{\delta g(T_i)}{\delta g(x)}\frac{\partial \mathcal{S}[\mathbf{g}]}{\partial g_i} = \sum_i \delta(T_i-x)\frac{\partial \mathcal{S}[\mathbf{g}]}{\partial g_i}.$$

The delta functions reduce the integrations over $\int dx$ to sums over the bond variables Σ_i. In particular, the evolution operator W_g reduces, on $\mathcal{S}[\mathbf{g}]$ to an ordinary partial differential operator with respect to the bond variables g_i. In symbols

$$W_g \mathcal{S}[\mathbf{g}] = \left[-\frac{1}{2}\int_{t_*}^{T} dx \int_{t_*}^{T} dx' \sum_{i,j=1}^{N} G_{ij}\delta(T_i-x)\delta(T_j-x')\frac{\partial^2}{\partial g_i \partial g_j} \right.$$

$$\left. -\int_{t_*}^{T} dx \sum_{i=1}^{N} \beta_i \delta(T_i-x)\frac{\partial}{\partial g_i} \right]\mathcal{S}[\mathbf{g}] \equiv W\mathcal{S}[\mathbf{g}]. \qquad (26)$$

Hence, from Eqs. (26) and (25)

$$W = -\frac{1}{2}\sum_{i,j=1}^{N} G_{ij}\frac{\partial^2}{\partial g_i \partial g_j} - \sum_{i=1}^{N} \beta_i \frac{\partial}{\partial g_i},$$

$$G_{ij} = G(T_i,T_j), \quad \beta_i = \beta(T_i) = \frac{1}{2}G_{ii}. \qquad (27)$$

Note G_{ij} is the forward bond propagator that appears in the price of a coupon bond European option [5].

Recall that the coupon bond, at time t_*, is as follows:

$$\mathcal{B}(t_*,T) = \sum_i c_i B(t_*,T_i), \quad B(t_*,T_i) = e^{-\int_{t_*}^{T_i} f(t_*,x)dx}.$$

In the Hamiltonian formulation, the coupon bond is written as an element of the state of $\mathcal{V}_*$, namely, $|\mathcal{B}(t_*,T)\rangle$ and

$$\langle \mathbf{g}|\mathcal{B}(t_*,T)\rangle = \mathcal{B}(t_*,T)[\mathbf{g}] = \sum_i c_i e^{-\int_{t_*}^{T_i} f(x)dx} \equiv \sum_i c_i e^{-g_i},$$

$$g_i \equiv g(T_i) = \int_{t_*}^{T_i} dx f(x),$$

$$\int_{\mathbf{g}} |\mathbf{g}\rangle\langle \mathbf{g}| = \mathcal{I}, \quad \int_{\mathbf{g}} \equiv \prod_i^{N} \int_{-\infty}^{+\infty} dg_i,$$

where $\mathbf{g} = (g_1,g_2,\dots,g_N)$. A similar definition for $f_i \equiv \int_{t_*}^{T_i} dx f(t_0,x)$ is given in Eq. (29); the g_i's will usually be employed as intermediate states with the f_i's reserved to

specify the initial (market) value of the forward interest rates.

V. COUPON BOND EUROPEAN OPTION PRICE

The price of the coupon bond European option has been obtained, to $O(\sigma^4)$ in Ref. [5], using Feynman perturbation expansion. An empirical study of the result applied to the Libor market swaption price, retaining only the second order $O(\sigma^2)$ term of the expansion, showed that the result has a root mean square error of less than 3% [7]. The European option price is rederived using the Hamiltonian formulation as a warm up for the much more complex derivation of the barrier option.

The European coupon bond option price, at time t_0, is given by

$$C_E(t_0, t_*, T, K) = B(t_0, t_*)\langle \mathbf{f}|e^{-W}|\mathcal{P}_*\rangle. \qquad (28)$$

The dual vector $\langle \mathbf{f}|$ is contructed from the initial value of $f(t_0, x)$ as follows:

$$f_i \equiv \int_{t_*}^{T_i} dx f(t_0, x), \quad \langle \mathbf{f}| \equiv \langle f_1, f_2, \dots, f_N| = \prod_{i=1}^{N} \langle f_i|. \quad (29)$$

The eigenfunctions of W are given by

$$\psi_{\mathbf{p}}[\mathbf{f}] = e^{i\sum_i p_i f_i} \equiv e^{i\mathbf{p}\cdot\mathbf{f}} = \langle \mathbf{f}|\mathbf{p}\rangle,$$

$$W\psi_{\mathbf{p}}[\mathbf{f}] = \langle \mathbf{f}|W|\mathbf{p}\rangle = \left(S - i\sum_{i=1}^{N}\beta_i p_i\right)\psi_{\mathbf{p}}[\mathbf{f}],$$

$$S = -\frac{1}{2}\sum_{ij=1}^{N} p_i G_{ij} p_j. \qquad (30)$$

The orthogonality and completeness of the eigenfunctions are given by

$$\int_{\mathbf{p}}|\mathbf{p}\rangle\langle\mathbf{p}| = \mathcal{I}, \quad |\mathbf{p}\rangle = |p_1, p_2, \dots, p_N\rangle,$$

$$\int_{\mathbf{p}} \equiv \prod_{i}^{N}\int_{-\infty}^{+\infty}\frac{dp_i}{2\pi}, \quad \langle\mathbf{p}|\mathbf{p}'\rangle = \prod_{i}[2\pi\delta(p_i - p_i')].$$

Inserting, in Eq. (28), the completeness equation for both the $|\mathbf{p}\rangle$ and $|\mathbf{g}\rangle$ yields

$$C_E(t_0, t_*, T, K) = B(t_0, t_*)\int_{\mathbf{p},\mathbf{g}}\langle\mathbf{f}|e^{-W}|\mathbf{p}\rangle\langle\mathbf{p}|\mathbf{g}\rangle\langle\mathbf{g}|\mathcal{P}_*\rangle$$

$$= B(t_0, t_*)\int_{\mathbf{p},\mathbf{g}} e^S e^{i(\mathbf{f}+\beta-\mathbf{g})\mathbf{p}}\mathcal{P}_*[\mathbf{g}].$$

The expansion coefficients, from Eqs. (15)–(17) are given as follows:

$$C_0^E = E[1] = \int_{\mathbf{p},\mathbf{g}} e^{S_0^E}e^{i(\mathbf{f}+\beta-\mathbf{g})\mathbf{p}}$$

$$= \int_{\mathbf{p}} e^{S_0^E}e^{i(\mathbf{f}+\beta)\mathbf{p}}\prod_{i}[2\pi\delta(p_i)], \quad S_0^E = 0,$$

$$\Rightarrow C_0^E = 1.$$

The second coefficient is given by

$$C_1^E = E[V] = \sum_I J_I E[e^{-g_I + f_I} - 1]$$

$$= \sum_I J_I \int_{\mathbf{p},\mathbf{g}} e^{S_1^E}e^{i(\mathbf{f}+\beta-\mathbf{g})\mathbf{p}}[e^{-g_I+f_I} - 1]$$

$$= \sum_I J_I\left[\int_{\mathbf{p},\mathbf{g}} e^{S_1^E}e^{-i\mathbf{g}\cdot\mathbf{p}}e^{-g_I-\beta_I} - C_0\right]$$

$$= \sum_I J_I\left\{\int_{\mathbf{p}} e^{S_1^E}e^{-\beta_I}\prod_{i\neq I}[2\pi\delta(p_i)]2\pi\delta(p_I + i) - 1\right\}$$

$$\Rightarrow C_1^E = \sum_I J_I[e^{(1/2)G_{II}}e^{-\beta_I} - 1] = 0,$$

$$S_1^E = \frac{1}{2}G_{II}$$

since, from Eqs. (9) and (25) $\beta_I = G_{II}/2$. The third coefficient is given by

$$C_2^E = E[V^2] = \sum_{IK} J_I J_K E[(e^{-g_I+f_I} - 1)(e^{-g_I+f_I} - 1)]$$

$$= \sum_{IK} J_I J_K \int_{\mathbf{p},\mathbf{g}} e^{S_2^E}e^{i(\mathbf{f}+\beta-\mathbf{g})\mathbf{p}}[e^{-g_I+f_I} - 1][e^{-g_K+f_K} - 1]$$

$$= \sum_{IK} J_I J_K \int_{\mathbf{p},\mathbf{g}} e^{S_2^E}e^{-i\mathbf{g}\cdot\mathbf{p}}[e^{-g_I-g_K-\beta_I-\beta_K} - e^{-g_I-\beta_I}$$

$$- e^{-g_K-\beta_I} + 1].$$

Using the result for C_0^E and C_1^E yields

$$C_2^E = \sum_{IK} J_I J_K\left\{\int_{\mathbf{p},\mathbf{g}} e^{S_2^E}e^{-i\mathbf{g}\cdot\mathbf{p}}e^{-g_I-g_K-\beta_I-\beta_K} - 1\right\}$$

$$= \sum_{IK} J_I J_K\left\{\int_{\mathbf{p}} e^{S_2^E}e^{-\beta_I-\beta_K}\prod_{i\neq I,K}[2\pi\delta(p_i)]\right.$$

$$\left.\times 2\pi\delta(p_I + i)2\pi\delta(p_K + i) - 1\right\}.$$

The momentum delta functions yield $p_I = -i = p_K$ with all other components $p_i = 0; i \neq I, K$. Hence, from Eq. (30),

$$S_2^E = -\frac{1}{2}\sum_{ij=1}^{N} p_i G_{ij} p_j = G_{IK} + \frac{1}{2}G_{II} + \frac{1}{2}G_{KK}. \qquad (31)$$

Combining Eq. (31) with $\beta_I = G_{II}/2$ yields

$$C_2^E = \sum_{IK} J_I J_K[e^{G_{IK}} - 1] \simeq \sum_{IK} J_I J_K G_{IK} + O(\sigma^3). \qquad (32)$$

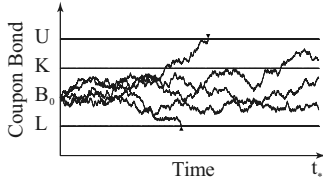The result agrees, as expected, with the earlier result obtained by path integration [5].

FIG. 5. The payoff function for a barrier option, with strike price K and maturity time t_*. The initial coupon bond value $B_0 \in [L, U]$, that is, is within the barrier, and the option is knocked out if the coupon bond, at any time before maturity at t_*, goes outside the barrier. Only trajectories that lie within the barrier contribute to the price of the barrier option.

VI. BARRIER OPTIONS

Barrier options, on maturing, have the same payoff as European options, namely, $|\mathcal{P}_*\rangle$, with the additional condition that, for the double barrier, the option is terminated with zero payoff if, at any instant before the option matures, the price of the underlying coupon bond $\mathcal{B}(t_*, T)$ exceeds a certain maximum value, say U or falls below a minimum value, say L. The price of the coupon bond barrier option is hence given by

$$\mathcal{C}_B(t_0, t_*, T, K) = B(t_0, t_*) \langle \mathbf{f} | e^{-W} | \mathcal{P}_* \rangle|_{\text{barrier}},$$

$$L \leq \sum_{i=1}^{N} c_i F(t, t_*, T_i) \leq U, \quad t_0 \leq t \leq t_*. \quad (33)$$

Figure 5 shows the payoff function of the coupon bond barrier option from time t_0 until it matures at time t_*. A coupon bond, which is allowed to have its price only in the range of $[L, U]$, is identical to a particle whose position is confined in the interval $[a, b]$. In quantum mechanics, the particle's position is confined by putting the particle inside an infinite potential well such that the potential $U(x)$ is infinite when x is outside the interval $[a, b]$, as shown in Fig. 6. A particle permanently trapped inside a potential well is described by eigenfunctions that are zero for all values of the position outside the interval $[a, b]$.

The barrier condition is incorporated into the barrier option pricing formula, similar to a quantum particle being confined to a potential well, by imposing the appropriate bound-
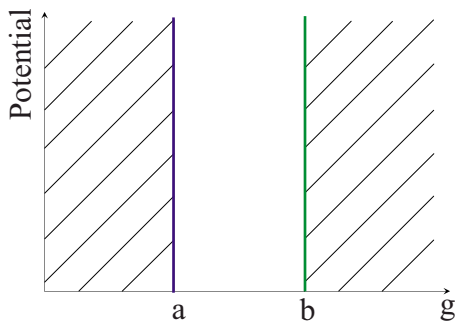

FIG. 6. (Color online) The potential well that confines the position of a quantum particle to be inside the interval $[a, b]$.

ary conditions on the eigenfunctions of the W operator [3]. In particular, for the coupon bond option, let $\langle \mathbf{g} | \psi_n \rangle \equiv \psi_n[\mathbf{g}]$ be a complete set of eigenfuctions of W that satisfy

$$W|\psi_k\rangle = -S_k|\psi_k\rangle, \quad \sum_k |\psi_k\rangle\langle\psi_k| = \mathcal{I}.$$

The barrier option is realized by imposing the following two boundary conditions on the eigenfunctions:

$$\mathcal{B} \equiv \sum_i c_i e^{-g_i},$$

$$\text{BC:} \psi_k[\mathbf{g}] = 0 \quad \text{for } \mathcal{B} \geq L \text{ and } \mathcal{B} \leq U.$$

The price of the barrier option is then given by

$$\mathcal{C}_B(t_0, t_*, T, K) = B(t_0, t_*) \langle \mathbf{f} | e^{-W} | \mathcal{P}_* \rangle|_{\text{barrier}}$$

$$= B(t_0, t_*) \sum_k \langle \mathbf{f} | e^{-W} | \psi_k \rangle \langle \psi_k | \mathcal{P}_* \rangle$$

$$= B(t_0, t_*) \sum_k e^{S_k} \psi_k[\mathbf{f}] \langle \psi_k | \mathcal{P}_* \rangle. \quad (34)$$

VII. ZERO COUPON BOND BARRIER OPTION

The coupon bond barrier option is calculated based on the eigenfunction realization of the barriers and illustrates, for the simplest case, the specific features of the barrier. The payoff function is given by

$$\mathcal{P}_* = (e^{-g} - K)_+,$$

$$g = \int_{t_*}^{T} dx f(t_*, x), \quad e^{-g} \in [e^{-b}, e^{-a}] \Rightarrow g \in [a, b].$$

In the Hamiltonian formulation, the barrier option is given, from Eq. (34), by

$$\mathcal{C}_B(t_0, t_*, K) = B(t_0, t_*) \sum_k e^{S_k} \psi_k[f] \langle \psi_k | \mathcal{P}_* \rangle,$$

$$W\psi_k(g) = -S_k \psi_k(g), \quad \psi_k(a) = 0 = \psi_k(b), \quad a \leq g \leq b,$$

$$g = f = \int_{t_*}^{T} dy f(t_0, y), \quad \psi_k[f] = \langle f | \psi_k \rangle.$$

The evolution operator, given by Eqs. (27) and (25), simplifies to

$$W = -\frac{1}{2} G \frac{\partial^2}{\partial g^2} - \beta \frac{\partial}{\partial g},$$

$$G = \int_{t_*}^{T} dy \int_{t_*}^{T} dy' \int_{t_0}^{t_*} dt M(y, y'; t),$$

$$\beta = \int_{t_*}^{T} dy \int_{t_0}^{t_*} dt\, \alpha(t,y) = \frac{1}{2}G.$$

To incorporate the barrier one solves the Schrödinger eigenfunction equation

$$(W + U)\psi_k(g) = -S_k\psi_k(g),$$

where the potential, as shown in Fig. 6, is given by

$$U(g) = \begin{cases} 0, & a \le g \le b, \\ \infty, & g \le a, g \ge b. \end{cases}$$

Consider the following ansatz for the eigenfunction:

$$\langle g|\psi_k\rangle = \psi_k(g) \sim \begin{cases} e^{(ik-\gamma)(g-a)}, & a \le g \le b, \\ 0, & g \le a, g \ge b, \end{cases}$$

$$W\psi_k(g) = \left[\frac{1}{2}G(k^2 - \gamma^2) + ik(G\gamma - \beta) + \beta\gamma\right]\psi_k(g).$$

γ is chosen to eliminate the term linear in k, which then yields two degenrate solutions $\psi_{\pm k}$. Hence,

$$\gamma = \frac{\beta}{G}, \quad \psi_{\pm k}(g) \sim e^{i(\pm k + i\gamma)(g-a)},$$

$$W\psi_{\pm k}(g) = -S_k\psi_{\pm k}(g), \quad S_k = -\frac{1}{2}\left(Gk^2 + \frac{\beta^2}{G}\right). \quad (35)$$

To impose the barrier option boundary conditons one superposes the degenerate solutions $\psi_{\pm k}(g)$ to obtain [12]

$$\langle g|\psi_k\rangle = \sqrt{\frac{2}{b-a}}e^{-\gamma g}\sin k(g-a),$$

$$\langle \psi_k|g\rangle = \sqrt{\frac{2}{b-a}}e^{\gamma g}\sin k(g-a),$$

$$k \equiv k_n = \frac{\pi n}{(b-a)}, \quad n = 1, 2, \dots, +\infty, \quad \psi_k(a) = 0 = \psi_k(b).$$

Hence, the eigenfunction for all values of g can be written as follows [13]:

$$\Psi_k(g) = [\theta(g-a) - \theta(g-b)]\psi_k(g), \quad -\infty \le g \le +\infty, \tag{36}$$

where

$$\theta(g-a) - \theta(g-b) = \begin{cases} 1, & a < g < b, \\ 0, & g \le a, \quad g \ge b. \end{cases}$$

The eigenfunctions are orthogonal since

$$\langle \psi_{k_n}|\psi_{k_{n'}}\rangle = \frac{2}{b-a}\int_a^b \sin k_n(g-a)\sin k_{n'}(g-a)dg = \delta_{n-n'}. \tag{37}$$

The Poisson summation formula, given by

$$\sum_{n=-\infty}^{\infty} e^{2\pi inx} = \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{38}$$

yields the following:

$$\sum_k \langle g|\psi_k\rangle\langle\psi_k|g'\rangle = \frac{2}{b-a}e^{-\gamma(g-g')}$$

$$\times\sum_{n=1}^{\infty}\sin k_n(g-a)\sin k_n(g'-a)$$

$$= \frac{1}{2(b-a)}e^{-\gamma(g-g')}$$

$$\times\sum_{n=-\infty}^{\infty}\left[\exp\left(\frac{in\pi}{b-a}(g-g')\right)\right.$$

$$\left. -\exp\left(\frac{in\pi}{b-a}(g+g'-2a)\right)\right]$$

$$= \frac{1}{2(b-a)}e^{-\gamma(g-g')}$$

$$\times\sum_{n=-\infty}^{\infty}\left[\delta\left(\frac{g-g'}{2(b-a)}-n\right)\right.$$

$$\left. -\delta\left(\frac{g+g'-2a}{2(b-a)}-n\right)\right]$$

$$= \delta(g-g') \text{ since } a < g, g' < b. \tag{39}$$

Hence, the eigenfuctions satisfy the completeness equation given by

$$\sum_k |\psi_k\rangle\langle\psi_k| = \mathcal{I}. \tag{40}$$

Inserting both the completeness equation given in Eq. (40) and the completeness equation for the coordinate eigenstate given by

$$\int_{-\infty}^{+\infty} dg|g\rangle\langle g| = \mathcal{I}$$

into the expression for the barrier option given in Eq. (33) yields, for $f = \int_{t_*}^{T} dy f(t_0, y) \in [a, b]$, the following exact price:

$$\frac{\mathcal{C}_B(t_0, t_*, K)}{B(t_0, t_*)} = \langle\mathbf{f}|e^{-W}|\mathcal{P}_*\rangle|_{\text{barrier}}$$

$$= \sum_{k=1}^{\infty}\int_{-\infty}^{+\infty}dg\langle f|e^{-W}|\psi_k\rangle\langle\psi_k|g\rangle\langle g|\mathcal{P}_*\rangle$$

$$= \frac{2}{(b-a)}\int_a^b dg\sum_{k=1}^{\infty}e^{S_k}e^{\gamma(g-f)}$$

$$\times\sin k_n(f-a)\sin k_n(g-a)\mathcal{P}_*(g)$$

$$= \frac{e^{-\beta^2/2G}}{2(b-a)}\int_a^b dg\sum_{n=-\infty}^{+\infty}e^{\gamma(g-f)}e^{-(1/2)Gk_n^2}$$

$$\times[e^{ik_n(f-g)} - e^{ik_n(f+g-2a)}]\mathcal{P}_*(g),$$

where we recall that $k_n = \pi n/(b-a)$.

The price of barrier option has another representation that is useful for $G \sim 0$. The Poisson summation formula given by Eq. (38) yields

$$\frac{1}{2(b-a)}\sum_{n=-\infty}^{+\infty}\exp\left\{-\frac{1}{2}G\left(\frac{\pi n}{b-a}\right)^2 + i\frac{\pi n\phi}{b-a}\right\}$$

$$= \sqrt{\frac{1}{2\pi G}}\sum_{n=-\infty}^{+\infty}\exp-\frac{1}{2G}[\phi - 2(b-a)n]^2. \quad (41)$$

Hence, since $\gamma = \beta/G$, the exact price of the zero coupon barrier option is given by

$$\mathcal{C}_B(t_0,t_*,K) = B(t_0,t_*)e^{-\beta^2/2G}\int_a^b dg\, e^{(\beta/G)(g-f)}$$

$$\times \mathcal{Q}[g,f;G;a,b]\mathcal{P}_*(g), \quad (42)$$

where the barrier function, from Eq. (41), is given by

$$\mathcal{Q}[g,f;G;a,b] = \frac{1}{\sqrt{2\pi G}}\sum_{n=-\infty}^{+\infty}\left[\exp\left(-\frac{1}{2G}[f-g\right.\right.$$

$$\left.-2(b-a)n]^2\right) - \exp\left(-\frac{1}{2G}[f+g-2a\right.$$

$$\left.\left.-2(b-a)n]^2\right)\right]. \quad (43)$$

The barrier function is plotted in Figs. 7 and 8. The price of a zero coupon bond barrier option is identical with a similar option for equity since both have only one independent degree of freedom.

The zero coupon bond European option is a special case of the double barrier option, for which $a \to -\infty$ and $b \to +\infty$; only the $n=0$ term in the first sum survives and yields

$$\mathcal{Q}_E[g,f;G;a,b] = \sqrt{\frac{1}{2\pi G}}\exp\left\{-\frac{1}{2G}(f-g)^2\right\} \quad (44)$$

and hence
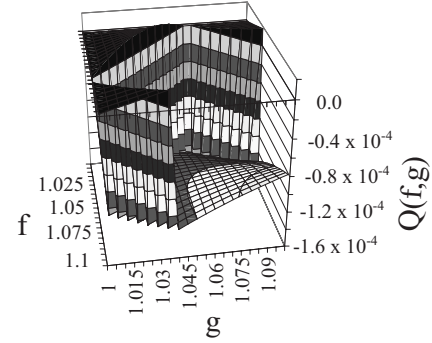
$$\mathcal{C}_B(t_0,t_*,K) \to B(t_0,t_*)\frac{e^{-\beta^2/2G}}{\sqrt{2\pi G}}\int_{-\infty}^{+\infty}dg\, e^{\beta/G(g-f)}$$

$$\times e^{-1/2G(f-g)^2}(e^{-g}-K)_+$$

$$= B(t_0,t_*)\frac{1}{\sqrt{2\pi G}}\int_{-\infty}^{+\infty}dg\, e^{-1/2G(f+\beta-g)^2}(e^{-g}-K)_+$$

$$= C_E(t_0,t_*,K).$$

As expected, on removing the barriers, the barrier option reduces to the zero coupon bond European option.

An option with only a single barrier on the right at $f=b$, shown in Fig. 9(b), is obtained by taking $a \to -\infty$ and is given by $n=0$ for the first term and $n=1$ as follows:

Barrier function Q(f,g) [a=1, b=1.1, G=0.3]
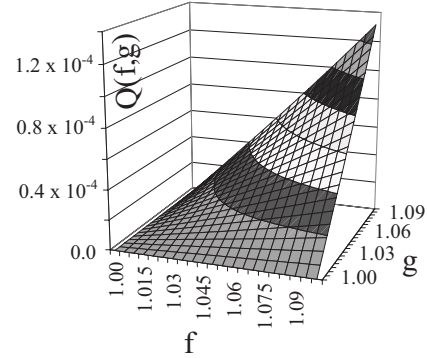

Barrier function Q(f,g) [a=1, b=1.1, G=0.6]

FIG. 7. Barrier knock-out option for values of the payoff below a minimum value of L or a maximum value of U; shaded portion shows the forbidden regions.

$$\mathcal{Q}_a[g,f;G;a;\infty] = \sqrt{\frac{1}{2\pi G}}\left[\exp\left\{-\frac{1}{2G}(f-g)^2\right\}\right.$$

$$\left.-\exp\left\{-\frac{1}{2G}(f-g-2a)^2\right\}\right].$$

Similarly, single barrier on the left at the position $f=a$, as shown in Fig. 9(a), is given by taking $b \to +\infty$ and is given by $n=0$ for the first term and $n=0$ for the second term

Barrier as a function of f and G

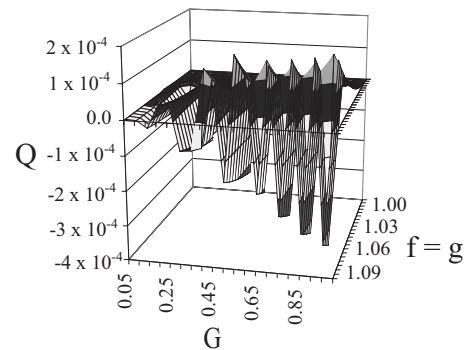FIG. 8. Barrier function $\mathcal{Q}[g,f;G;a,b]$. As seen from the two diagrams, for $G<0.3$ the barrier function is very irregular as a function of f, g, smoothening out for $G>0.6$.
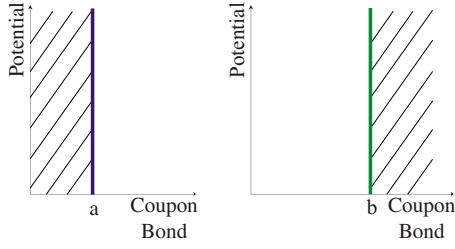
FIG. 9. (Color online) Barrier function $\mathcal{Q}[g,f;G;a,b]$, with $f = g$ and for different values of G. As a function of G the barrier function is seen be quite irregular, having large variations for small changes in G.

$$\mathcal{Q}_b[g,f;G;-\infty,b] = \sqrt{\frac{1}{2\pi G}} \left[\exp\left\{ -\frac{1}{2G}(f-g)^2 \right\} - \exp\left\{ -\frac{1}{2G}(f-g-2b)^2 \right\} \right].$$

VIII. LINEARIZATION OF THE BARRIER

The exact price of the zero coupon bond barrier option could be obtained because the barrier is a linear function of the forward interest rates. In contrast, the case of the coupon bond is far more complicated due to the nonlinear nature of the barrier. Recall, from Eqs. (27) and (34), the evolution operator and eigenfunctions for the barrier option are given by

$$W = -\frac{1}{2} \sum_{i,j=1}^{N} G_{ij} \frac{\partial^2}{\partial g_i \partial g_j} - \sum_{i=1}^{N} \beta_i \frac{\partial}{\partial g_i},$$

$$W\psi_k[\mathbf{g}] = -S_k \psi_n[\mathbf{g}], \quad \text{BC}: \psi_k[\mathbf{g}] = 0$$

$$\text{for } \sum_I c_I e^{-g_I} \geq L \text{ and } \sum_I c_I e^{-g_I} \leq U. \quad (45)$$

The nonlinear barrier function is shown in Fig. 10. The evolution operator W is similar to the Laplacian operator in N
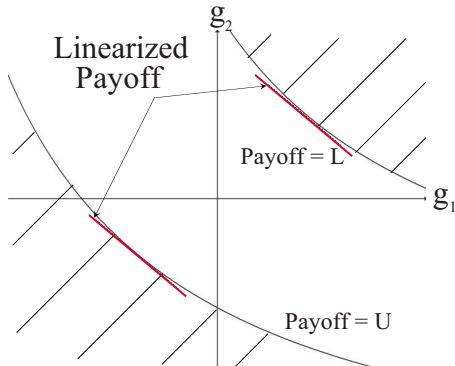

FIG. 10. (Color online) The nonlinear barrier condition. The coupon bond trajectories, from time t_0 to t_*, take values only in the (unshaded) permitted domain—lying in-between U and L; all other trajectories do not contribute to the option price.

dimensions, and the barriers define two $(N-1)$-dimensional subspaces via the nonlinear equations $\Sigma_I c_I e^{-g_I} = L$ and $\Sigma_I c_I e^{-g_I} = U$; the boundary condition that the eigenfuctions are zero on these subspaces in turn implies that one needs to solve for the eigenfunctions of the W operator that are zero on the nontrivial and nonlinear subspaces—in general, an intractable problem. For these reasons, analytically finding the exact price of a coupon bond barrier option is, in principle, almost impossible.

Due to the specific form of the coupon bond price an approximate solution for the bond barrier option can be found that is leading order in the forward interest rates' volatility $\sigma(t,x)$; this may be all one needs in practice since a detailed empirical analysis of the European option price for the coupon bond shows that the leading term in σ is orders of magnitude more important than the next order terms [7].

A small value of forward interest rate volatility σ implies that the fluctuations of the coupon bond about its initial value are of $O(\sigma)$; this was the reason that the initial value of the coupon bond was subtracted from the payoff function as in Eq. (10). Hence, a similar subtraction of the initial coupon bond price should yield $O(\sigma)$ fluctuations for the payoff function.

One needs to find the leading term in the barrier option constraint and linearize the barrier constraint about the leading term. To find the leading order term consider the following combination:

$$g_I - f_I - \beta_I = \int_{t_*}^{T_I} dx [f(t_*,x) - f(t_0,x)] - \beta_I. \quad (46)$$

The variance follows from the following representation of the forward interest rates

$$f(t_*,x) = f(t_0,x) + \int_{t_0}^{t_*} dt\, \alpha(t,x) + \int_{t_0}^{t_*} dt\, \sigma(t,x) A(t,x),$$

$$g_I = f_I + \beta_I + \int_{t_0}^{t_*} dt\, \sigma(t,x) A(t,x)$$

$$\text{with} \quad E[A(t,x)A(t',x')] = \delta(t-t') D(x,x';t)$$

and is given by

$$E[(g_I - f_I - \beta_I)^2] = \int_{t_0}^{t_*} dt \int_{t_*}^{T_I} dx \int_{t_*}^{T_I} dx'\, M(x,x';t)$$

$$= G_{II} \sim O(\sigma^2).$$

The calculation above shows that all the fluctuations of the random quantity $g_I - f_I - \beta_I$ is of $O(\sigma)$; the β_I term needs to subtracted to account for the drift of g_I from time t_0 to maturity time t_*. One has the following linearization of the barrier condition

$$\sum_I c_I e^{-g_I} = \sum_I c_I e^{-f_I - \beta_I} e^{-(g_I - f_I - \beta_I)} = \sum_I d_I e^{-(g_I - f_I - \beta_I)}$$
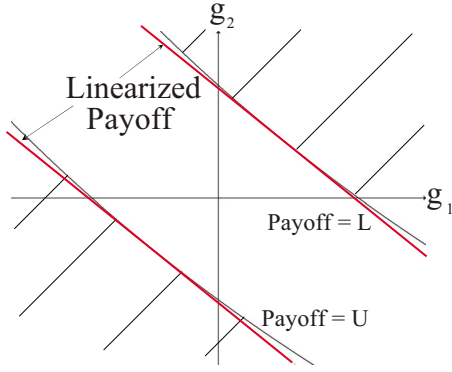
$$\text{where } d_I \equiv c_I e^{-f_I - \beta_I},$$

FIG. 11. (Color online) Linearized barrier constraint.

$$\Rightarrow \mathcal{B}(t_0, T) = \sum_I c_I e^{-g_I} \simeq \sum_I d_I (1 + f_I + \beta_I) - \sum_I d_I g_I + O(\sigma^2).$$
$$(47)$$

Define the new barrier limits (shown in Fig. 12)

$$a = \sum_I d_I (1 + f_I + \beta_I) - U,$$

$$b = \sum_I d_I (1 + f_I + \beta_I) - L.$$

The new linearized barrier conditions are now defined by

$$\mathrm{BC}: \psi_k[\mathbf{g}] = 0 \text{ for } gd \geq a \text{ and } gd \leq b,$$

$$gd \equiv \sum_I g_I d_I. \tag{48}$$

Figure 11 shows that for small values of the bond variables the linearization of the the barrier function yields a good approximation; the coefficients d_I are chosen to ensure that the linearization takes into account of the leading value of the coupon bond—since it is only around this leading value are the fluctuations going to be small. Note that the linearization of the barrier cannot be systematically improved by, say, expanding the barrier to quadratic or higher terms in the bond variables g_i; the reason being that there are no systematic techniques that can generate the eigenfunctions on nontrivial domains that result from including the higher order nonlinear terms (see Fig. 12).

IX. OVERCOMPLETE BARRIER EIGENFUNCTIONS

The linearized barrier constraints can be implemented via eigenfuctions of W in a manner similar to the one used for the zero coupon bond barrier option. There is, however, an additional feature of coupon bonds that is not present for the zero coupon case, namely, for the coupon bond the linear sum of all the bond variables, namely, $gd = \sum_I d_I g_I$, needs to be constrained. A symmetric combination all of the coordinates implies that a change of variables from the g_i; $i = 1, 2, \ldots, N$ to another set of N variables will, in general, not place all the g_i's on an equal footing.
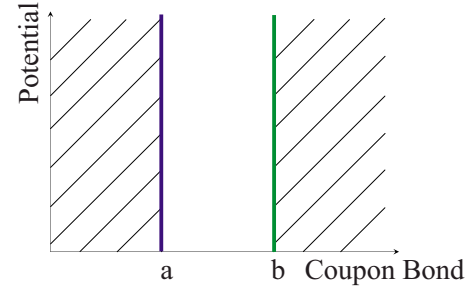
FIG. 12. (Color online) A knock-out double barrier option realized by a potential for the payoff that is infinite for values greater than U and for values below a minimum value of L.

One way out of this conundrum is to increase the space of complete eigenfuctions by including another crucial eigenfunction of gd, namely, the eigenfunction that carries the barrier condition. This leads to more eigenfunctions than are required for providing a complete basis for the state space in the completeness equation and is compensated by adding a constraint—analogous to the construction of coherent states in quantum mechanics.

In analagoy with the zero coupon bond case given in Eq. (36), consider the following ansatz for the coupon bond eigenfunctions:

$$\Psi_{\mathbf{p},k}(\mathbf{g}) = e^{i\mathbf{p}\cdot\mathbf{g}} \psi_k(gd), \quad -\infty \leq g_i \leq +\infty,$$

$$\psi_k(gd) \sim [\theta(gd - a) - \theta(gd - b)] e^{i(k+i\gamma)(gd-\alpha)},$$
$$-\infty \leq gd \leq +\infty. \tag{49}$$

Note the eigenfunctions $e^{i\mathbf{p}\cdot\mathbf{g}}$ form a complete basis, as given in Eq. (31), and including the eigenfunction $\psi_k(gd)$ makes the eigenfunctions $\Psi_{\mathbf{p},k}(\mathbf{g})$ overcomplete.

The operator W acting on the eigenfunctions, using a notation for later convenience, yields the following:

$$W[e^{i\mathbf{p}\cdot\mathbf{g}} \psi_k(gd)] = (-S - i\beta\mathbf{p})[e^{i\mathbf{p}\cdot\mathbf{g}} \psi_k(gd)],$$

$$-S = \frac{1}{2}pGp + \frac{v^2}{2}(k + i\gamma)^2 + (k + i\gamma)pGd - ik\beta d + \gamma\beta d,$$

$$pGp \equiv \sum_{I,J=1}^N p_I G_{IJ} p_J, \quad pGd \equiv \sum_{I,J=1}^N p_I G_{IJ} d_J,$$

$$v^2 \equiv dGd = \sum_{I,J=1}^N d_I G_{IJ} d_J, \quad \beta d = \sum_{I=1}^N \beta_I d_I. \tag{50}$$

As in the zero coupon bond case, γ is chosen to eliminate the terms in S that are linear in k so that one can obtain two degenrate solutions $\psi_{\pm k}(gd)$. This yields

$$\gamma = \frac{1}{v^2}(\beta d + ipGd) \tag{51}$$

and which, in turn, gives the following result:

$$-S = \frac{1}{2}pGp + \frac{v^2}{2}k^2 + \frac{1}{v^2}[(\beta d)^2 - (pGc)^2 + i(\beta d)pGd].$$
$$(52)$$

To impose the barrier option boundary conditions, following the case of the zero coupon bond, one superposes the degenerate solutions $\psi_{\pm k}(gd)$ to obtain, for $a \leq gd \leq b$, the following [14]:

$$\langle \mathbf{g}|\psi_k \rangle = \sqrt{\frac{2}{b-a}} e^{-\gamma gd} \sin k(gd - a),$$

$$\langle \psi_k|\mathbf{g} \rangle = \sqrt{\frac{2}{b-a}} e^{\gamma gd} \sin k(gd - a),$$

$$k \equiv k_n = \frac{\pi n}{(b-a)}, \quad n = 1, 2, \ldots, +\infty,$$

$$\psi_k(gd)\big|_{gd=a} = 0 = \psi_k(gd)\big|_{gd=b}. \qquad (53)$$

Unlike the case of the zero coupon barrier option, the constraint equation for the coupon bond involves N variables; the constraint is realized by the following representation:

$$\theta(gd - a) - \theta(gd - b) = \int_a^b dh\, \delta(h - gd)$$

$$= \int_a^b dh \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} e^{i\xi(h-gd)}$$

$$\equiv \int_{h,\xi} e^{i\xi(h-gd)}. \qquad (54)$$

The completeness equation is given by

$$\mathcal{I} = \left(\sum_i d_i\right) \int_{\mathbf{p}} 2\pi \delta\left(\sum_i p_i\right) \sum_k |\Psi_{\mathbf{p},k}\rangle\langle\Psi_{\mathbf{p},k}|$$

$$= \left(\sum_i d_i\right) \int_{\mathbf{p}} 2\pi \delta\left(\sum_i p_i\right) \sum_k |\psi_k\rangle|\mathbf{p}\rangle\langle\mathbf{p}|\langle\psi_k|. \qquad (55)$$

There is an extra Σ_k sum over the additional eigenfunctions $|\psi_k\rangle$ due to overcompleteness. Unlike Eq. (31), the constraint $\delta(\Sigma_i p_i)$ has been introduced in the completeness equation to compensate for the overcomplete set of eigenfunctions.

To prove the completeness equation, consider the following:

$$\langle \mathbf{f}|\mathcal{I}|\mathbf{g}\rangle = \left(\sum_i d_i\right) \int_{\mathbf{p}} 2\pi \delta\left(\sum_i p_i\right) \sum_k \langle \mathbf{f}|\mathbf{p}\rangle\langle\mathbf{f}|\psi_k\rangle\langle\psi_k|\mathbf{g}\rangle\langle\mathbf{p}|\mathbf{g}\rangle$$

$$= \left(\sum_i d_i\right) \int_{\mathbf{p}} 2\pi \delta\left(\sum_i p_i\right) e^{i\mathbf{p}(\mathbf{f}-\mathbf{g})} \mathcal{F},$$

$$\mathcal{F} = \frac{2}{b-a}\sum_k \int_{h,\xi,h',\xi'} e^{i\xi(h-fd)} e^{i\xi'(h'-gd)} e^{-\gamma(fd-gd)}$$

$$\times \sin\{k(fd - a)\sin[k(gd - a)]\}. \qquad (56)$$

The bond state $|\mathbf{g}\rangle$ is taken to be unrestricted. The initial

value of the coupon bond is within the barrier, that is, $a \leq fd \leq b$ as shown in Fig. 5, and which implies that $\int_{h,\xi} e^{i\xi(h-fd)} = 1$; from Eq. (39)

$$\mathcal{F} = \frac{2e^{-\gamma(fd-gd)}}{b-a}\sum_k \sin\{k(fd - a)\sin[k(gd - a)]\}$$

$$\times \int_{h,\xi} e^{i\xi(h-gd)}$$

$$= \delta(fd - gd) \int_{h,\xi} e^{i\xi(h-gd)}. \qquad (57)$$

Since $2\pi\delta(\Sigma_i p_i) = \int_{-\infty}^{+\infty} d\zeta \exp(i\zeta\Sigma_i p_i)$, Eqs. (56) and (57) yield

$$\langle \mathbf{f}|\mathcal{I}|\mathbf{g}\rangle = \left(\sum_i d_i\right) \int_{-\infty}^{+\infty} d\zeta \prod_{i=1}^N \delta(f_i - g_i + \zeta)$$

$$\times \int_{h,\xi} e^{i\xi(h-gd)} \delta(fd - gd)$$

$$= \left(\sum_i d_i\right) \int_{-\infty}^{+\infty} d\zeta \prod_{i=1}^N \delta(f_i - g_i + \zeta)$$

$$\times \int_{h,\xi} e^{i\xi(h-gd)} \delta\left(\zeta\sum_i d_i\right)$$

$$= \int_{h,\xi} e^{i\xi(h-gd)} \prod_{i=1}^N \delta(f_i - g_i) = \prod_{i=1}^N \delta(f_i - g_i), \quad (58)$$

where the last equation above is a consequence of $a \leq fd \leq b$.

Hence, Eq. (58) confirms that the overcomplete set of eigenfunctions in Eq. (55) yield the correct completeness equation. For the zero coupon bond barrier options, Eq. (55) reduces to the one given in Eq. (40) since the constraint $\delta(p)$ removes the extra eigenfunctions e^{ipg}.

The completeness equation requires only that the bond vector $\langle \mathbf{f}|$ must satisfy the condition that $a \leq fd \leq b$, leaving the bond vector $|\mathbf{g}\rangle$ completely free; this more general result is required when the completeness equation (55) is used for evaluating the barrier option price.

X. COUPON BOND BARRIER OPTION PRICE

The approximate price of the coupon bond barrier option is given by Eqs. (34) and (55) as follows [15]:

$$\frac{C_B(t_0, t_*, T, K)}{B(t_0, t_*)} = \langle \mathbf{f}|e^{-W}|\mathcal{P}_*\rangle\big|_{\text{barrier}}$$

$$= \left(\sum_i d_i\right) \int_{\mathbf{p}} 2\pi \delta\left(\sum_i p_i\right) \sum_k \langle \mathbf{f}|e^{-W}|\Psi_{\mathbf{p},k}\rangle$$

$$\times \langle \Psi_{\mathbf{p},k}|\mathcal{P}_*\rangle$$

$$= \left(\sum_i d_i\right) \int_{\mathbf{p},\mathbf{g}} 2\pi \delta\left(\sum_i p_i\right) \sum_k \langle \mathbf{f}|e^{-W}|\Psi_{\mathbf{p},k}\rangle$$

$$\times \langle \Psi_{\mathbf{p},k}|\mathbf{g}\rangle\langle\mathbf{g}|\mathcal{P}_*\rangle \qquad (59)$$

$$\Rightarrow \frac{\mathcal{C}_B(t_0,t_*,T,K)}{B(t_0,t_*)} \equiv \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]\mathcal{P}_*[\mathbf{g}], \qquad (60)$$

where the last equation defines the pricing kernel $\mathcal{K}$ for the barrier option. The linearized boundary conditions for the barrier option eigenfunctions, from Eq. (48), are given by

$$\text{BC:} \quad \Psi_{\mathbf{p},k}[\mathbf{g}] = 0 \text{ for } gd \geq b \text{ and } gd \leq a.$$

Consider an initial value of the coupon bond, as shown in Fig. 5, that lies within the barrier; the linearized approximation implies that $a \leq fd \leq b$; Eqs. (49), (50), and (52)–(54) yield the following:

$$\langle \mathbf{f}|e^{-W}|\Psi_{\mathbf{p},k}\rangle = e^{S+i\beta\mathbf{p}}\langle \mathbf{f}|\Psi_{\mathbf{p},k}\rangle$$

$$= \sqrt{\frac{2}{b-a}} e^{S+i(\beta+\mathbf{f})\mathbf{p}-\gamma fd} \sin k(fd-a),$$

$$S = -\frac{1}{2}pGp - \frac{v^2}{2}k^2 - \frac{1}{v^2}[(\beta d)^2 - (pGc)^2 + i(\beta d)pGd],$$

$$\langle \Psi_{\mathbf{p},k}|\mathbf{g}\rangle = \sqrt{\frac{2}{b-a}} e^{-i\mathbf{g}p+\gamma gd} \sin k(gd-a) \int_{\xi,h} e^{i\xi(h-gd)},$$

$$\mathcal{P}_*[\mathbf{g}] = \left[\sum_I c_I e^{-g_I} - K \right]_+.$$

Equations (60) and (13) yield the following expansion for the barrier option:

$$\frac{\mathcal{C}_B(t_0,t_*,T,K)}{B(t_0,t_*)} = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]\mathcal{P}_*[\mathbf{g}]$$

$$\simeq \int_{Q,\eta} e^{-i\eta Q}(F+Q-K)_+$$

$$\times \left[C_0^B + iC_1^B\eta - \frac{1}{2}C_2^B\eta^2 + O(\sigma^3) \right].$$

A detailed derivation of the coefficients is given in Appendix B and yields the following results:

$$C_0^B = E[1] = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]$$

$$= e^{S_0^B} \int_a^b dh \exp\left\{ \frac{1}{v^2}\beta d(h-fd) \right\}$$

$$\times \mathcal{Q}[h,fd;v^2;a,b],$$

$$S_0^B = -\frac{1}{2v^2}(\beta d)^2.$$

The second coefficient is given by

$$C_I^B = E[e^{-g_I+f_I}] = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]e^{-g_I+f_I}$$

$$= e^{S_I^B} \int_a^b dh \exp\left\{ \frac{1}{v^2}\left(\beta d - \sum_J G_{IJ}d_J\right)(h-fd) \right\}$$

$$\times \mathcal{Q}[h,fd;v^2;a,b],$$

$$S_I^B = -\frac{1}{2v^2}\left(\beta d - \sum_J G_{IJ}d_J\right)^2.$$

The third coefficient is given by

$$C_{IK}^B = E[e^{-g_I-g_K+f_I+f_K}]$$

$$= \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]e^{-g_I-g_K+f_I+f_K}$$

$$= e^{S_{IK}^B} \int_a^b dh \exp\left\{ \frac{1}{v^2}\left(\beta d - \sum_J G_{IJ}d_J - \sum_J G_{KJ}d_J\right) \right.$$

$$\times (h-fd) \bigg\} \mathcal{Q}[h,fd;v^2;a,b],$$

$$S_{IK}^B = G_{IK} - \frac{1}{2v^2}\left(\beta d - \sum_J G_{IJ}d_J - \sum_J G_{KJ}d_J\right)^2.$$

Recall that the barrier function is given by Eq. (43).

To extract the perturbative expansion of the option price to $O(\sigma)$ from the coefficients C_0, C_1, and C_2 the leading order term for the barrier option has to be isolated. On inspecting the coefficients, it is clear that in fact C_0 is a term of $O(1)$, with C_1 and C_2 being of order $O(\sigma)$ and $O(\sigma^2)$, respectively.

From the results for the coefficients given in this section one has

$$\frac{\mathcal{C}_B(t_0,t_*,T,K)}{B(t_0,t_*)} = e^{-(1/2)v^2(\beta d)^2} \int_a^b dh \exp\left\{ \frac{1}{v^2}\beta d(h-fd) \right\}$$

$$\times \mathcal{Q}[h,fd;v^2;a,b] \int_{Q,\eta} e^{-iQ\eta}\left[1 + i\eta D_1 \right.$$

$$\left. - \frac{1}{2}\eta^2 D_2 + O(\sigma^3) \right](Q+F-K)_+. \qquad (61)$$

Equation (61) is one of the most important results of this paper, namely, the barrier option price is the integral of two factor, namely, the function $\exp\{-\frac{1}{2v^2}(\beta d)^2 + \frac{1}{v^2}\beta d(h-fd)\}\mathcal{Q}[h,fd;v^2;a,b]$ that encodes the properties of the barrier and the other factor $\int_{Q,\xi} e^{-iQ\eta}[1+i\eta D_1 - \frac{1}{2}\eta^2 D_2 + O(\sigma^3)] \times (Q+F-K)_+$ that encodes the properties of the payoff function. Each factor has been evaluated approximately and one

can improve the price of the barrier option by improving the approximation for each of these factors.

The coefficients D_1, D_2 are given by

$$D_1 = \sum_I J_I [e^{S_I^B - S_0^B} e^{-(1/v^2)\Sigma_J G_{IJ}d_J)(h-fd)} - 1]$$

$$\simeq -\frac{1}{v^2} \left[\sum_{I,J} J_I G_{IJ} d_J \right] (h - fd) + \Gamma,$$

$$\Gamma = \frac{\beta d}{v^2} \sum_{I,J} J_I G_{IJ} d_J + \frac{1}{2v^4} \sum_I J_I \left(\sum_J G_{IJ} d_J \right)^2 [(h-fd)^2 - v^2]$$

$$+ O(\sigma^3),$$

$$D_2 = \sum_{I,K} J_I J_K [e^{S_{IK}^B - S_0^B} e^{-(1/v^2)(\Sigma_J G_{IJ}d_J + \Sigma_J G_{KJ}d_J)(h-fd)}$$

$$- e^{S_I^B - S_0^B} e^{-(1/v^2)\Sigma_J G_{IJ}d_J(h-fd)}$$

$$- e^{S_K^B - S_0^B} e^{-(1/v^2)\Sigma_J G_{KJ}d_J)(h-fd)} + 1]$$

$$\simeq \sum_{I,K} J_I J_K G_{IK} - \frac{1}{v^2} \left[\sum_{IJ} J_I G_{IJ} d_J \right]^2$$

$$+ \frac{1}{v^4} \left[\sum_{IJ} J_I G_{IJ} d_J \right]^2 (h-fd)^2 + O(\sigma^3).$$

The results for D_1 and D_2 yield

$$D_2 - D_1^2 \simeq \sum_{I,K} J_I J_K G_{IK} - \frac{1}{v^2} \left[\sum_{IJ} J_I G_{IJ} d_J \right]^2$$

$$+ \Gamma \left[\frac{2}{v^2} \sum_{I,J} J_I G_{IJ} d_J (h-fd) - \Gamma \right] + O(\sigma^3).$$

$$(62)$$

Collecting all the terms yields, from Eq. (18), the following main result for the approximate price of the barrier option:

$$\frac{C_B(t_0, t_*, T, K)}{B(t_0, t_*)} = \frac{e^{-(1/2v^2)(\beta d)^2}}{\sqrt{2\pi}} \int_a^b dh \, \exp\left\{ \frac{1}{v^2} \beta d(h - fd) \right\}$$

$$\times \mathcal{Q}[h, fd; v^2; a, b] I(X) \sqrt{D_2 - D_1^2} + O(\sigma^3),$$

$$X = (K - F - D_1)/\sqrt{D_2 - D_1^2}.$$

From Eq. (19) one has $I(X) = 1 + O(X)$ and this yields the leading order price of the barrier option

$$\frac{C_B(t_0, t_*, K)}{B(t_0, t_*)} = \frac{e^{-(1/2v^2)(\beta d)^2}}{\sqrt{2\pi}} \int_a^b dh \, \exp\left\{ \frac{1}{v^2} \beta d(h - fd) \right\}$$

$$\times \mathcal{Q}[h, fd; v^2; a, b] \sqrt{D_2 - D_1^2} + O(X).$$

XI. LIMITING CASES OF THE BARRIER OPTION

The price for the European option and zero coupon barrier option were derived in Secs. V and VII. The result obtained

for the coupon bond barrier option has the expected limiting behavior.

A. The one factor HJM model

The HJM model [8,4] of the forward interest rates is widely used in finance and is a special case of the quantum finance model. In the HJM approach all the forward interest rates, in the language of quantum finance, are exactly correlated and this implies that $D(x, x'; t) \rightarrow 1$ and hence $M(x, x', t) = \sigma(t, x)\sigma(t, x')$. Furthermore, in the one factor HJM model the volatility function to have an exponential form given by $\sigma(t, x) = \sigma_0 e^{-\lambda(x-t)}$.

Taking the HJM limit of the bond correlator yields

$$G_{ij} = \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x, x', t) \rightarrow G_{ij}^{\text{HJM}}$$

$$= \sigma_0^2 \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \, e^{-\lambda(x-t)} \int_{t_*}^{T_j} dx' \, e^{-\lambda(x'-t)}$$

$$= \sigma_R^2 Y_i Y_j,$$

$$Y_i \equiv Y(t_*, T_i) = \frac{1}{\lambda} [1 - e^{-\lambda(T_i - t_*)}],$$

$$\sigma_R^2 = \frac{\sigma_0^2}{2\lambda} [1 - e^{-2\lambda(t_* - t_0)}].$$

For the HJM case, the first two terms for the coupon bond barrier option in Eq. (62) cancel and yield the following HJM limit:

$$[D_1]_{\text{HJM}} = -\frac{JY}{Yd}(h - fd) + \Gamma_{\text{HJM}},$$

$$[D_2 - D_1^2]_{\text{HJM}} = \Gamma_{\text{HJM}} \left[2\frac{JY}{Yd}(h - fd) - \Gamma_{\text{HJM}} \right],$$

$$\Gamma_{\text{HJM}} = \frac{\beta d}{Yd} JY + \frac{1}{2}\frac{JY^2}{(Yd)^2}[(h - fd)^2 - \sigma_R^2(Yd)^2],$$

$$Yd = \sum_i Y_i d_i, \quad JY = \sum_i J_i Y_i, \quad JY^2 = \sum_i J_i Y_i^2.$$

B. Zero coupon bond barrier option

The coupon bond barrier option reduces to the zero coupon case when only the final coupon is nonzero, hence $c_N = 1$ and $c_i = 0$, $i \neq N$. The barrier functions also reduce to $d_N = 1$ and $d_i = 0$, $i \neq N$ and yields

$$\beta d \rightarrow \beta, \quad v^2 = \sum_{IJ} d_I G_{IJ} d_J \rightarrow G_{NN} = G,$$

$$\sum_{IJ} G_{IJ} d_J \rightarrow G_{NN} = G, \quad G_{IK} \rightarrow G_{NN} = G.$$

Hence, using the martingale that yields $\beta = G/2$, one has

$$S_0^B \rightarrow -\frac{\beta^2}{2G}, \quad S_I^B \rightarrow -\frac{(\beta - G)^2}{2G} = -\frac{\beta^2}{2G},$$

$$S_{IK}^B \rightarrow G - \frac{(\beta - 2G)^2}{2G} = -\frac{\beta^2}{2G},$$

$$\mathcal{Q}[h, fd; v^2; a, b] \rightarrow \mathcal{Q}[h, f; G; a, b],$$

$$\mathcal{P}_* \rightarrow [J(e^{-(g-f)} - 1) + F - K]_+, \quad J = e^{-f} = F.$$

Collecting the results above yields, from Eq. (61), the following zero coupon limit of the coupon bond barrier option:

$$\mathcal{C}_B(t_0, t_*, T, K) \rightarrow B(t_0, t_*) \int_{Q, \eta} e^{-i\eta Q} (F + Q - K)_+ \left[C_0^B + i C_1^B \eta \right.$$

$$\left. - \frac{1}{2} C_2^B \eta^2 + O(\sigma^3) \right]$$

$$= e^{-(\beta^2/2G)} \int_a^b dg \, \mathcal{Q}[g, f; G; a, b] e^{\beta/G(g-f)}$$

$$\times \left(1 + i\eta[Je^{-(g-f)} - 1] \right.$$

$$\left. - \frac{1}{2} \eta^2 [J^2 e^{-2(g-f)} - 2Je^{-(g-f)} + 1] \right).$$

The result above yields that

$$\frac{\mathcal{C}_B(t_0, t_*, T, K)}{B(t_0, t_*)} \rightarrow e^{-\beta^2/2G} \int_a^b dg \, e^{(\beta/G)(g-f)} \mathcal{Q}[g, f; G; a, b] \mathcal{P}_*(g)$$

$$+ O(\sigma^3),$$

which is the expected approximation of the exact result given in Eq. (42).

C. Coupon bond European option limit

Consider the limit of $a \rightarrow -\infty$ and $b \rightarrow \infty$; the function $\mathcal{Q}$ reduces to a single term, namely,

$$\mathcal{Q}[h, fd; v^2; a, b] \rightarrow \frac{1}{\sqrt{2\pi v^2}} \exp -\frac{1}{2v^2} [h - fd]^2,$$

$$\int_a^b dh \rightarrow \int_{-\infty}^{+\infty} dh. \tag{63}$$

Performing the Gaussian integrations over h yields the expected that the barrier option become equal to the European option; in particular,

$$C_0^B \rightarrow 1 = C_0^E,$$

$$C_1^B = \sum_I J_I(C_I^B - C_0^B) \rightarrow \sum_I J_I(1 - 1) = 0 = C_1^E,$$

$$C_2^B = \sum_{I,J} J_I J_K (C_{IK}^B - C_I^B - C_J^B + C_0^B)$$

$$\rightarrow \sum_{I,J} J_I J_K (e^{G_{IK}} - 1)$$

$$= C_2^E.$$

The perturbative barrier option is equal to the European option—in the limit of $a \rightarrow -\infty$ and $b \rightarrow +\infty$— due to the following results:

$$e^{-(1/2v^2)(\beta d)^2} \int_a^b dh \, \exp\left\{ \frac{1}{v^2} \beta d(h - fd) \right\} \mathcal{Q}[h, fd; v^2; a, b] D_1$$

$$\simeq \frac{1}{4} \sum_I J_I(\beta d)^2 = 0 + O(\sigma^4),$$

$$e^{-(1/2v^2)(\beta d)^2} \int_a^b dh \, \exp\left\{ \frac{1}{v^2} \beta d(h - fd) \right\} \mathcal{Q}[h, fd; v^2; a, b] D_2$$

$$\simeq G_{IK} + \frac{1}{2v^4} \sum_{IK} J_I J_K (\beta d)^2 \sum_J G_{IJ} d_J \sum_J G_{KJ} d_J$$

$$= G_{IK} + O(\sigma^4).$$

XII. CONCLUSIONS

The Hamiltionian formulation of the quantum field theory of forward interest rates provides an efficient computation tool for analyzing the coupon bond European and barrier options. The earlier result for the European coupon bond option and swaption [5] is seen to emerge in a straightforward manner in the Hamiltonian approach.

The zero coupon barrier option price was obtained exactly by imposing the constraint of the barrier on the eigenfunctions of the Hamiltonian or, more accurately, of the time integrated Hamiltonian, namely, the evolution operator. The evolution operator was expressed in terms of the "bond variables" that are more natural for studying coupon bonds.

The computation of the coupon bond barrier option price turned out to be more difficult than the zero coupon bond case for two reasons. First, because many different zero coupon bonds taken together constitute a coupon bond and second, because the barrier on the coupon bond imposes a nonlinear constraint on the forward interest rates. It was shown that, under very general conditions, the linearized payoff function yields the leading contribution to the coupon bond barrier option price. An overcomplete set of eigenfunctions of the evolution operator was used for imposing the linearized barrier on the evolution of the bond variables.

Equation (61) shows that the entire calculation for the barrier option factorizes into two connected and nontrivial components, one factor reflecting the properties of the barrier conditon and the other that of the payoff function. This feature of the barrier options seems to be very general and could prove useful in analyzing options with more complex barriers and payoffs. The coupon bond barrier option price is an analytic function of the initial forward rate curve, the strike

TABLE I. The drift and cross-term of p_i's with d_i's for the different expansion coefficients.

	$ipGd = i\Sigma_{IJ} p_I G_{IJ} d_J$	$\gamma = \dfrac{1}{v^2}(\beta d + ipGd)$
C_0^B:	0	$\dfrac{1}{v^2}\beta d$
C_I^B:	$\dfrac{v^2}{\Sigma_I d_I} - \Sigma_J G_{IJ} d_J$	$\dfrac{2}{v^2}\left(\beta d + \dfrac{v^2}{\Sigma_I d_I} - \Sigma_J G_{IJ} d_J\right)$
C_{IK}^B:	$2\dfrac{v^2}{\Sigma_I d_I} - \Sigma_J G_{IJ} d_J - \Sigma_J G_{KJ} d_J$	$\dfrac{2}{v^2}\left(\beta d + 2\dfrac{v^2}{\Sigma_I d_I} - \Sigma_J G_{IJ} d_J - \Sigma_J G_{KJ} d_J\right)$

price, the duration of option and the barrier; hence all the hedging parameters can be (approximately) evaluated analytically from the results obtained.

Quantum finance provides a flexibile and powerful framework for the study of interest rate options as illustrated by the computation of the coupon bond barrier and European option price. Taking the bond variables to be exactly correlated, as is the case with the HJM model, results in systematic errors in the pricing and hedging of coupon bond European (and barrier) options [7]. In contrast, the nontrivial correlations of the bond variables are parsimoniously encoded in quantum finance by the imperfect correlator G_{IJ} and the barrier condition is efficiently modeled by eigenfunctions of the pricing Hamiltionian.

APPENDIX A: STATE SPACE OF COUPON BOND OPTION

In the Hamiltonian formulation the option price at time t_0 is given by propagating the payoff function $|\mathcal{P}_*\rangle$, maturing at time t_* backwards in time, as given in Eq. (2), to present time t_0, discounted by the numeraire $B(t_0, t_*)$. Hence, in terms of the basis state at t_0 given by $\langle f^{(0)}|[\equiv \Pi_{t_0 \leq x \leq t_0 + T_{FR}} \langle f(x)|]$, the European coupon bond option price is given by

$$C_E(t_0, t_*, T, K) = B(t_0, t_*) E[(\mathcal{B}(t_*, T) - K)_+]$$

$$= B(t_0, t_*) \langle f^{(0)}| T\left\{\exp - \int_{t_0}^{t_*} dt H(t)\right\} |\mathcal{P}_*\rangle.$$

(A1)

Note time is flowing backwards. Using the completeness equation $\int Df_* |f_*\rangle\langle f_*| = \mathcal{I}$,

$$\langle f^{(0)}| T\left\{\exp - \int_{t_0}^{t_*} dt H(t)\right\} |\mathcal{P}_*\rangle$$

$$= \int Df_* \langle f^{(0)}| T\left\{\exp - \int_{t_0}^{t_*} dt \mathcal{H}(t)\right\} |f_*\rangle\langle f_*|\mathcal{P}_*\rangle.$$

(A2)

Equation (A2) can be simplified further. For time $t \in [t_0, t_*]$, the nonzero overlap of the basis state $\langle f_*|[\equiv \Pi_{t_* \leq x \leq t_* + T_{FR}} \langle f(x)|]$ with $|\mathcal{P}_*\rangle$ is only in the interval $x \in [t_*, T]$; this in turn implies that the state vector $\langle f^{(0)}|$ is projected to the state vector $\langle f_*^{(0)}| \equiv \langle f_*| = \Pi_{t_* \leq x \leq T} \langle f(x)|$.

The domain $\mathcal{R}$ required for computing the matrix element in Eq. (A2) is given in Fig. 4. The domain $\mathcal{R}$ has the important feature that the state spaces $\mathcal{V}_t$ for all $t \in [t_0, t_*]$ are fixed in time and are all identical, spanned by variables $f(x), x \in [t_*, T]$; moreover, on domain $\mathcal{R}$, the Hamiltonian commutes for different times $[\mathcal{H}_*(t), \mathcal{H}_*(t')] = 0$ since only the coefficients $M(x, x'; t)$ and $\alpha(t, x)$ are time dependent.

For these reasons the time ordering $\mathcal{T}$ in Eq. (A2) is no longer necessary, and can be removed. $\mathcal{H}_*(t)$ is consistently restricted to the domain $\mathcal{R}$ by limiting the range of $x \in [t_*, T]$. Hence

$$H_*(t)|_\mathcal{R} = -\frac{1}{2}\int_{t_*}^T dx dx'\, \sigma(t, x) D(x, x'; t) \sigma(t, x') \frac{\delta^2}{\delta f(x)\delta f(x')}$$

$$- \int_{t_*}^T dx\, \alpha(t, x) \frac{\delta}{\delta f(x)}.$$

Since there is no longer any time ordering for the Hamiltonian, the operator driving the option price is given by a new time integrated operator, namely, the evolution operator, given by

$$W \equiv \int_{t_0}^{t_*} dt H_*(t)|_\mathcal{R},$$

TABLE II. Value of quadratic function of the p_i's for the different expansion coefficients.

	$pGp = \Sigma_{IJ} p_I G_{IJ} p_J$
C_0^B:	0
C_I^B:	$-\dfrac{v^2}{\Sigma_I d_I} + \dfrac{2}{\Sigma_I d_I} \Sigma_J G_{IJ} d_J - G_{II}$
C_{IK}^B:	$-2G_{IK} - G_{II} - G_{KK} - 4\left(\dfrac{v^2}{\Sigma_i d_i}\right)^2 + \dfrac{4}{\Sigma_i d_i}(\Sigma_i G_{iI} d_i + \Sigma_i G_{iK} d_i)$

TABLE III. The value of the "action" for the different expansion coefficients.

	$S_p = -\dfrac{1}{2} pGp - \dfrac{1}{v^2}[(\beta d)^2 - (pGd)^2 + i(\beta d)pGd]$
$S_0^B =$	$-\dfrac{1}{2v^2}(\beta d)^2$
$S_I^B =$	$-\dfrac{1}{2v^2}(\beta d - \Sigma_J G_{IJ} d_J)^2$
$S_{IK}^B =$	$G_{IK} - \dfrac{1}{2v^2}(\beta d - \Sigma_J G_{IJ} d_J - \Sigma_J G_{IK} d_J)^2$

$$W = -\frac{1}{2} \int_{t_*}^{T} dx dx' M(x,x') \frac{\delta^2}{\delta f(x)\delta f(x')} - \int_{t_*}^{T} \alpha(x) \frac{\delta}{\delta f(x)},$$

$$\alpha(x) = \int_{t_0}^{t_*} dt \alpha(t,x),$$

$$M(x,x') = \int_{t_0}^{t_*} dt \sigma(t,x) D(x,x';t)\sigma(t,x'). \quad (A3)$$

Hence, the option price is given

$$\frac{C_E(t_0,t_*,T,K)}{B(t_0,t_*)} = \langle f^{(0)}| \mathcal{T}\left\{\exp - \int_{t_0}^{t_*} dt H(t)\right\}|\mathcal{P}_*\rangle.$$

$$= \langle f_*|e^{-W}|\mathcal{P}_*\rangle \quad (A4)$$

The option price is completely determined by the matrix element of e^{-W} taken between two vectors $\langle f_*|$ and $|\mathcal{P}_*\rangle$ that both belong to the same state space $\mathcal{V}_*$. For barrier options, the price is determined by the same matrix element with the barrier being imposed on the eigenfunctions of W.

If the discounting factor, instead of $B(t_0,t_*)$, was taken to be $\exp[-\int_{t_0}^{t_*} dt r(t)]$—where $r(t) = f(t,t)$—the domain for evaluating the matrix element in Eq. (A2) is the trapezoidal domain given in Fig. 13. Since the discounting factor $r(t)$ is a random quantity it needs to be taken inside the time ordering and the discounting factor thus extends the nonzero over-
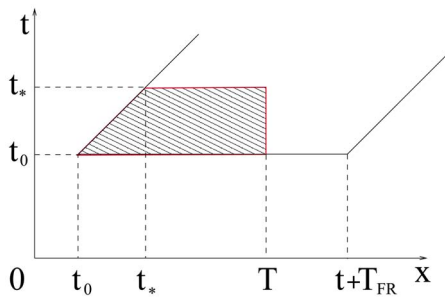

FIG. 13. (Color online) The domain of forward interest rates that determines the price of the coupon bond option if the discouting was done with $\exp[-\int_{t_0}^{t_*} dt r(t)]$.

lap of the basis state $\langle f_t|$ with $\exp[-\int_{t_0}^{t_*} dt r(t)]|\mathcal{P}_*\rangle$ to the interval $x \in [t,T]$. This would in turn imply that the time ordering symbol $\mathcal{T}$ cannot be ignored since the underlying state space and Hamiltonian are now time dependent; one would need to do a separate (and more complicated) calculation for each $t \in [t_0,t_*]$.

APPENDIX B: EVALUATION OF COEFFICIENTS OF BARRIER OPTION

The coefficients for the coupon bond barrier option are defined in Sec. X as follows:

$$C_0^B = E[1] = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}],$$

$$C_I^B = E[e^{-g_I + f_I}] = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]e^{-g_I + f_I},$$

$$C_{IK}^B = E[e^{-g_I - g_K + f_I + f_K}] = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]e^{-g_I - g_K + f_I + f_K}.$$

The initial conditions yield that $a \leq fd \leq b$ and Eqs. (49), (50), and (52)–(54) yield the following:

$$\mathcal{K}[\mathbf{f},\mathbf{g}] = \frac{2\sum_i d_i}{b-a} \sum_{k=1}^{\infty} \int_{\xi,h} \int_{\mathbf{p},\mathbf{g}} 2\pi\delta\left(\sum_i p_i\right)$$
$$\times e^{i\xi(h-gd)}e^{S+i(\beta+\mathbf{f})\mathbf{p}-i\mathbf{g}\cdot\mathbf{p}}e^{\gamma(gd-fd)}$$
$$\times \sin k(fd-a)\sin k(gd-a)$$

$$= \frac{\sum_I d_I}{2(b-a)} \sum_{k=-\infty}^{+\infty} \int_{\xi,h} \int_{\mathbf{p},\mathbf{g}} 2\pi\delta\left(\sum_i p_i\right)$$
$$\times e^{i\xi(h-gd)}e^{S+i(\beta+\mathbf{f})\mathbf{p}-i\mathbf{g}\cdot\mathbf{p}}e^{\gamma(gd-fd)}$$
$$\times [e^{-ik(gd-fd)} - e^{ik(gd+fd-2a)}]$$

and

$$S = -\frac{v^2}{2}k^2 + S_p,$$

$$S_p \equiv -\frac{1}{2}pGp - \frac{1}{v^2}[(\beta d)^2 - (pGc)^2 + i(\beta d)pGd],$$

$$\gamma = \frac{1}{v^2}(\beta d + ipGd), \quad v^2 = dGd.$$

Some details are given of the derivation of the coefficients C_{IK}^B; the other coefficients need a similar, but simpler, calculations. Shifting the variable $g_i \rightarrow g_i + f_i + \beta_i$ in the expression for C_{IK}^B and then performing the $\int_{\mathbf{g}}$ integrations yields the following:

$$C_{IK}^B = \int_{\mathbf{g}} \mathcal{K}[\mathbf{f},\mathbf{g}]e^{-g_I - g_K}$$

$$= \frac{\Sigma_i d_i}{2(b-a)} \sum_{k=-\infty}^{+\infty} e^{-(v^2/2)k^2} \int_{\xi,h} e^{i\xi(h - gd - fc - \beta d)}$$

$$\times \int_{\mathbf{p},\mathbf{g}} 2\pi\delta\left(\sum_i p_i\right) e^{S_p} e^{-i\mathbf{g}\cdot\mathbf{p}} e^{\gamma(gd + \beta d)}$$

$$\times [e^{-ik(gd + \beta d)} - e^{ik(gd + \beta d + 2fd - 2a)}]e^{-g_I - g_K}$$

$$= \frac{1}{2(b-a)} \sum_{k=-\infty}^{+\infty} e^{-(v^2/2)k^2} \int_{\xi,h} e^{i\xi(h - fd - \beta d)}[Ae^{-ik\beta d}$$

$$- Be^{ik(\beta d + 2fd - 2a)}], \tag{B1}$$

where

$$A = \left(\sum_i d_i\right)\int_{\mathbf{p}} e^{S_p + \gamma\beta d} 2\pi\delta\left(\sum_{i=1}^N p_i\right)\delta(p_K + \Lambda_A d_K - i)$$

$$\times \delta(p_I + \Lambda_A d_I - i) \prod_{i \neq I,K} \delta(p_i + \Lambda_A d_i),$$

$$\Lambda_A = i\gamma + \xi + k,$$

$$B = \left(\sum_i d_i\right)\int_{\mathbf{p}} e^{S_p + \gamma\beta d} 2\pi\delta\left(\sum_{i=1}^N p_i\right)$$

$$\times \delta(p_K + \Lambda_B d_K - i)\delta(p_I + \Lambda_B d_I - i) \prod_{i \neq I,K} \delta(p_i + \Lambda_B d_i),$$

$$\Lambda_B = i\gamma + \xi - k.$$

Dirac δ functions

One needs to explicitly solve the δ functions and fix the values of p_i, which is required for determining the action S_0 and the drift γ; furthermore, the explicit values of all the p_i's are required for carrying out the N integrations, namely, $\int_{\mathbf{p}}$ so as to evaluate A and B. The reason the δ functions apparently look intractable is because the function γ appears in the δ functions and γ is itself a function of the all the p_i's: the δ functions, in effect, yield a set of (apparently intractable) simultaneous equations for the p_i's.

However, by recursively solving the δ functions we have the rather remarkable result that the explicit value all the p_i's can be found in the following manner. Consider

$$\left(\sum_i d_i\right)2\pi\delta\left(\sum_{i=1}^N p_i\right)\delta(p_I + \Lambda_A d_I - i)\delta(p_K + \Lambda_A d_K - i)\prod_{i \neq I,K}\delta(p_i + \Lambda_A d_i)$$

$$= \left(\sum_i d_i\right)2\pi\delta\left(\Lambda_A \sum_{i=1}^N d_i - 2i\right)\delta(p_I + \Lambda_A d_I - i)\delta(p_K + \Lambda_A d_K - i)\prod_{i \neq I,K}\delta(p_i + \Lambda_A d_i)$$

$$= 2\pi\delta\left(\Lambda_A - \frac{2i}{\Sigma_i d_i}\right)\delta\left(p_I + \frac{2id_I}{\Sigma_i d_i} - i\right)\delta\left(p_K + \frac{2id_K}{\Sigma_i d_i} - i\right)\prod_{i \neq I,K}\delta\left(p_i + \frac{2id_i}{\Sigma_i d_i}\right). \tag{B2}$$

Equation (B2) uniquely fixes all the p_i's and one can hence perform the $\int_{\mathbf{p}}$ integration to explicitly obtain A. A similar result is obtained for B with Λ_B replacing Λ_A; note the contraints on the p_i's do not depend on Λ_A and hence the values of S_p and γ are the same for coefficients A and B.

There are $N+1$ δ function in Eq. (B2) and performing the $\int_{\mathbf{p}}$ integrations leaves over one δ function, which for the A term is given by $\delta(\Lambda_A - 2i/\Sigma_i d_i) = \delta(i\gamma + \xi + k - 2i/\Sigma_i d_i)$, where γ has been fixed by Eq. (B2). Using this δ function, and a similar one for the B term, to perform the ξ integration in Eq. (B1) yields the following:

$$C_{IK}^B = \frac{e^{S_{IK} + \gamma\beta d}}{2(b-a)} \sum_k \int_h e^{-(v^2/2)k^2}[e^{-i(k+i\gamma)(h - fd - \beta d)}e^{-ik\beta d}$$

$$- e^{-i(k-i\gamma)(h - fd - \beta d)}e^{ik(\beta d + 2fd - 2a)}]$$

$$= \frac{e^{S_{IK}}}{2(b-a)} \int_h e^{1/v^2(\beta d - \Sigma_J G_{IJ} d_J - \Sigma_J G_{IK} d_J)(h - fd)}$$

$$\times \sum_k e^{-v^2/2k^2}[e^{-ik(h - fd)} - e^{ik(h + fd - 2a)}]$$

$$= e^{S_{IK}} \int_a^b dh e^{1/v^2(\beta d - \Sigma_J G_{IJ} d_J - \Sigma_J G_{IK} d_J)(h - fd)}$$

$$\times \mathcal{Q}[h, fd, v^2; a, b],$$

where

$$S_{IK} = G_{IK} - \frac{1}{v^2}\left(\beta d - \sum_J G_{IJ} d_J - \sum_J G_{IK} d_J\right)^2$$

and $\mathcal{Q}[h, fd, v^2; a, b]$ is the barrier function given by Eq. (43).

The results for the different coefficients are obtained in the same manner as for C_{IK}^B; a remarkable result is that for all the coefficients the barrier function $\mathcal{Q}[h, fd, v^2; a, b]$ completely factorizes, leading to perturbative coefficients D_1 and D_2 that are evaluated in Sec. X.

The main results required for evaluating C_0^B, C_1^B and C_2^B are given in the Tables I–III; Tables I and II yield the values for the various terms in S_p for the different coefficients, with S_p itself given in Table III.

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[9] The term discounting is fundamental to finance. Consider the interest on a fixed deposit that is rolled over; this leads to an exponential compounding of the initial fixed deposit. Discounting, the inverse of the process of compounding, is the procedure that yields the present day value of a future prefixed sum of money.

[10] The fixed payoff $1 is assumed and is not written out explicitly.

[11] The matrix element of e^{-W_f} directly yields the price of the option; in contrast, in quantum mechanics, it is the absolute modulus squared of the matrix element that is an observable (physical) quantity.

[12] Note the evolution operator W is not Hermitian; hence, under the duality operation that takes the $|\psi_k\rangle$ to its dual vector $\langle\psi_k|$, the term $e^{-\gamma g}$ switches its sign to $e^{+\gamma g}$.

[13] Since $\partial\theta(g-a)/\partial g = \delta(g-a)$, it follows that $\Psi_k(g)$ is an eigenfunction of W with eigenvalue $-S_k$.

[14] Note the evolution operator W is not Hermitian; hence, under the duality operation that takes the $|\psi_k\rangle$ to its dual vector $\langle\psi_k|$, the term $e^{-\gamma g d}$ switches it's sign and goes to $e^{+\gamma g d}$.

[15] Equation (59) has been obtained by using $\int_\mathbf{g} |\mathbf{g}\rangle\langle\mathbf{g}| = \mathcal{I}$.